Twisted K-theory, groupoids and cohomology

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Quick review of twisted K-theory

Let M be a compact space. Donovan-Karoubi (1970) consider bundles $\mathcal{A} \longrightarrow M$ with fiber $M_n(\mathbb{C})$ (possibly with a $\mathbb{Z}/2\mathbb{Z}$-grading). Such bundles are classified up to stable isomorphism by the Brauer group

$$\mathrm{Br}(M) \cong H^3(M, \mathbb{Z})_{tors}.$$

The corresponding element $\delta(\mathcal{A})$ is called the Dixmier-Douady invariant of $\mathcal{A}$.

Definition 1. $K_{\mathcal{A}}(M) := K(C(M, \mathcal{A}))$.

It only depends up to non-canonical isomorphism from $\delta(\mathcal{A})$.

Initial motivation: if $\mathcal{A} = \mathrm{Cliff}(TM)$ then $\delta(\mathcal{A}) = 0$ if and only if M is $\mathrm{Spin}^{\mathbb{C}}$.

Generalization (J. Rosenberg): $H^3(M, \mathbb{Z})$ classifies bundles with fiber $\mathcal{K}(H)$.

Central extensions

If (U_i) is an open cover that trivializes $\mathcal{A}$, we get a cocycle $u_{ij} : U_{ij} \longrightarrow \mathrm{Aut}(\mathcal{K}(H)) = PU(H)$. Lift to $\tilde{u}_{ij} \in U(H)$, then

$$u_{ij}u_{jk} = c_{ijk}u_{ik}.$$

Then $c_{ijk} : U_{ijk} \longrightarrow \mathbb{T}$ is a cocycle.

Let $\Gamma = \amalg U_{ij}$. Then $\Gamma \rightrightarrows \Gamma_0$ is a groupoid with unit space $\Gamma_0 = \amalg U_i$ and product

$$(i, x, j)(j, x, k) = (i, x, k).$$

Let $\tilde{\Gamma} = \Gamma \times \mathbb{T}$ with the product

$$(i, x, j, \lambda)(j, x, k, \mu) = (i, x, k, \lambda\mu c_{ijk}(x)).$$

Then

$$\mathbb{T} \to \tilde{\Gamma} \to \Gamma$$

is a central extension of Γ. Moreover, Γ is Morita equivalent to M.

More generally, if $G \rightrightarrows G_0$ is a (proper Lie) groupoid, a twisting of G is given by

1) a central extension of groupoids

$$\mathbb{T} \to \tilde{\Gamma} \to \Gamma$$

2) a Morita equivalence between Γ and G.

Then $\mathcal{A} = C^*(\tilde{\Gamma})^{\mathbb{T}} = \overline{\{f \in C_c(\tilde{\Gamma})|\ f(\lambda g) = \lambda^{-1} f(g)\ \forall \lambda \in \mathbb{T},\ \forall g \in \tilde{\Gamma}\}}$ is, up to stabilization, the space of sections of an equivariant bundle over G_0 with fiber $\mathcal{K}(H)$.

In fact, Morita-equivalence classes of central extensions form a group, which is isomorphic to the Brauer group of G.

De Rham cohomology of groupoids

Let $\Gamma \rightrightarrows \Gamma_0$ be a groupoid. Let

$$\Gamma_n = \{(g_1, \ldots, g_n) \in \Gamma^n |\; g_1 \cdots g_n \text{ makes sense}\}.$$

Let $\partial : \Omega^q(\Gamma_p) \longrightarrow \Omega^q(\Gamma_{p+1})$,

$$\partial = \sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*,$$

where $\varepsilon_0(g_1, \ldots, g_{p+1}) = (g_2, \ldots, g_{p+1})$,
$\varepsilon_i(g_1, \ldots, g_{p+1}) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1})$,
$\varepsilon_{p+1}(g_1, \ldots, g_{p+1}) = (g_1, \ldots, g_p)$.
Then the double complex $(\Omega^q(\Gamma_p), d, \partial)$ is the de Rham complex of Γ.

Connections and curvings

Definition 2. Given an S^1-central extension $S^1 \to \tilde{X}_1 \to X_1 \rightrightarrows X_0$,

(i) a connection form $\theta \in \Omega^1(\tilde{X}_1)$ for the bundle $\tilde{X}_1 \to X_1$, such that $\tilde{\partial}\theta = 0$ is a *connection* on the S^1-central extension $S^1 \to \tilde{X}_1 \to X_1 \rightrightarrows X_0$;

(ii) Given θ, a 2-form $B \in \Omega^2(X_0)$, such that $d\theta = \tilde{\partial} B$ is a *curving*;

(iii) and given (θ, B), the 3-form $\Omega = dB \in H^0(X_\bullet, \Omega^3) \subset \Omega^3(X_0)$ is called the 3-*curvature*.

We have the following

Lemma 3. *Given an S^1-central extension $S^1 \to \tilde{X}_1 \xrightarrow{\pi} X_1 \rightrightarrows X_0$,*

(1) *the obstruction group to the existence of connections is $H^2(X_\bullet, \Omega^1)$;*

(2) *the obstruction group to the existence of curvings is $H^1(X_\bullet, \Omega^2)$.*

Connections exist if Γ is proper. Curvings exist if Γ is Morita equivalent to an étale groupoid.

For the rest of the talk, we will be concerned with twistings of $M \rtimes G$, where M is a compact manifold and G a compact Lie group acting on M.

If $\mathbb{T} \to \tilde{\Gamma} \to \Gamma$ is a central extension and Γ is Morita equivalent to $M \rtimes G$, then curvings may not exist, but one can remedy this in some sense.

We can assume that $\Gamma = p^*(M \rtimes G \rightrightarrows M)$, where $p : M' \to M$ is étale (for instance, $M' = \amalg U_{ij}$).

Let $N = M' \times G$ and $\sigma : N \to M$, $\sigma(x, g) = p(x)g$.

Let $H = N \times_M N$. Then $H \rightrightarrows N$ is G-equivariantly Morita equivalent to $M \rightrightarrows M$.

Since $H \to \Gamma$, $((x, g), (y, h)) \to (x, gh^{-1}, y)$, the twisting of $M \rtimes G$ is represented by a G-equivariant central extension

$$\mathbb{T} \to \tilde{H} \to H.$$

The Cartan complex

Recall that if G is a compact Lie group acting on a manifold M, then the *Cartan complex*

$$\Omega_G(M) = (S(\mathfrak{g}^*) \otimes \Omega(M))^G$$

with differential $d_G = d - \iota$, where

$$(d\omega)(X) = d(\omega(X))$$

$$(\iota\omega)(X) = \iota_X \omega(X)$$

computes the G-equivariant de Rham cohomology of M.

Connections and the Dixmier-Douady invariant

Any connection on $\mathbb{T} \to \tilde{\Gamma} \to \Gamma$ pulls-back to a G-invariant, G-basic connection $\theta \in \Omega^1(\tilde{H})$ for $\mathbb{T} \to \tilde{H} \to H$.

Let $B \in \Omega^2(N)^G$ be a curving.

Define $\Omega_G \in \Omega^3_G(M)$ by

$$\sigma^* \Omega_G = d_G B.$$

Then $[\Omega_G]$ is the Dixmier-Douady class of the central extension.

We will denote $\Omega_G = \alpha + \eta$, $\alpha \in \Omega^3(M)^G$, $\eta \in (\mathfrak{g}^* \otimes \Omega^1(M))^G$.

The inertia groupoid

Let Γ be a groupoid. Denote by $S\Gamma = \{g \in \Gamma |\ r(g) = s(g)\}$ the space of "loops". Then Γ acts by conjugation on the space $S\Gamma$. The groupoid $\Lambda\Gamma = S\Gamma \rtimes \Gamma$ is called the inertia groupoid of Γ.

Note that if $\Gamma = M \rtimes G$, then $M^g \rtimes G^g$ is isomorphic to a subgroupoid of $\Lambda\Gamma$.

Geometric transgression

Let $\mathbb{T} \to \tilde{\Gamma} \to \Gamma$ be a central extension, where Γ is Morita equivalent to $M \rtimes G$.

Let $L = \tilde{\Gamma} \times_{\mathbb{T}} \mathbb{C}$. It is a line bundle over the manifold Γ. Its restriction to $S\Gamma$ is a vector bundle over $\Lambda\Gamma \sim_{Mor.} \Lambda(M \rtimes G)$, so one gets vector bundles

$$L^{(g)} \to M^g$$

which are G^g-equivariant. They are endowed with canonical flat connections ∇^g (restrictions of any connection for the central extension).

Twisted cohomology: first approximation

For each $g \in G$, there is a twisted Cartan complex

$$\left(S((\mathfrak{g})^g)^*) \otimes \Omega(M^g, L^{(g)})\right)^{G^g}$$

with the differential $\nabla^g - \iota + 2\pi i(\alpha + \eta)$.

Twisted cohomology: approximative definition

For each $g \in G$, there is a twisted completed Cartan complex

$$(C_0^\infty(\mathfrak{g}^g, \Omega(M^g, L^{(g)}))^{G^g}$$

where C_0^∞ denotes germs at 0 of smooth functions, with the differential $\nabla^g - \iota + 2\pi i(\alpha + \eta)$.

(cf. Block-Getzler in the non-twisted case.)

The twisted cohomology is the cohomology of the complex consisting of equivariant sections of a sheaf over the space G (endowed with the adjoint action of G) whose stalk at $g \in G$ is $C_0^\infty(\mathfrak{g}^g, \Omega(M^g, L^{(g)}))$. It maps to the above complex.

We will use the notation $\mathcal{L}_X a = X \cdot a = \lim_{t \to 0} \frac{e^{tX} \cdot a - a}{t}$ for all $X \in \mathfrak{g}$.

Theorem 4. *Assume that $(\Omega^n)_{n \in \mathbb{N}}$ is a topological graded algebra endowed with an action of a compact Lie group G, $\rho : A \longrightarrow \Omega^0$ is a G-equivariant continuous morphism, ∇ is a G-equivariant derivation of degree 1 of $\Omega^\bullet$, $\iota : \mathfrak{g} \longrightarrow Der^{-1}(\Omega^\bullet)$ is a G-equivariant map satisfying*

(i) $\iota_X^2 = 0$ *for all* $X \in \mathfrak{g}$*;*

(ii) $\nabla\iota_X + \iota_X\nabla = \mathcal{L}_X$;
(iii) *there exists* $\Theta \in (\Omega^2)^G$ *such that* $\nabla^2 = [\Theta, \cdot]$ *and such that the elements* $\alpha = \nabla\Theta$ *and* $\eta_X = \iota_X\Theta$ *are central in* $\Omega^\bullet$, *for all* $X \in \mathfrak{g}$.

Moreover, we fix a central *element* $g \in G$ *and assume that* $((\bar{\Omega}^n)_{n\in\mathbb{N}}, d)$ *is a* G*-differential complex,* $\bar{\iota} : \mathfrak{g} \to \hom^{-1}(\bar{\Omega}^\bullet)$ *is a* G*-equivariant linear map and* $Tr_g : \Omega^n \to \bar{\Omega}^n$ *is a* G*-equivariant linear continuous map with the properties:*

(iv) $\bar{\iota}_X^2 = 0$;
(v) $d\bar{\iota}_X + \bar{\iota}_X d = \mathcal{L}_X$;
(vi) $Tr_g(\omega_1\omega_2) = (-1)^{|\omega_1|\,|\omega_2|}Tr_g((g^{-1}\omega_2)\omega_1)$ *for all* $\omega_i \in \Omega^\bullet$;
(vii) $d \circ Tr_g = Tr_g \circ \nabla$;
(viii) $\bar{\iota}_X \circ Tr_g = Tr_g \circ \iota_X$;
(ix) $\bar{\Omega}^\bullet$ *is a module over the subalgebra* Z *generated by* α *and the* η_X, *stable by* ∇ *and* ι_X, *so that for all* $\beta \in Z$ *and all* $\omega \in \bar{\Omega}^\bullet$, *we have* $Tr_g(\beta\omega) = \beta Tr_g\Omega$, $d(\beta\omega) = d\beta \cdot \omega + (-1)^{|\beta|}\beta \cdot d\omega$ *and* $\bar{\iota}_X(\beta\omega) = (\bar{\iota}_X\beta)\omega + (-1)^{|\beta|}\beta\iota_X\omega$.

Then, the (completed) Cartan complex $\bar{C}^\bullet_G(\bar{\Omega}) := C_0^\infty(\mathfrak{g}, \bar{\Omega}^\bullet)^G$ *is endowed with the differential* $d - \bar{\iota} + \alpha + \eta$*. Let then*

$$\tau(\varphi \otimes a_0 \otimes \cdots \otimes a_k)(X) = \varphi(ge^{-X}) \int_{\Delta_k} Tr_g(\rho(a_0)\nabla^{(t_1,X)}\rho(a_1)\cdots\nabla^{(t_k,X)}\rho(a_k))dt_1\cdots dt_k,$$

where $\Delta_k = \{(t_1, \dots, t_k)|\ 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$

and $\nabla^{(t,X)}a = e^{-t\Theta}\nabla(e^{tX}a)e^{t\Theta}$*.*

Then $\tau \circ (b + B) = (d - \bar{\iota} + \alpha + \eta) \circ \tau$*, thus* τ *induces a morphism*

$$\tau : HP_*^G(A) \to H^*(\bar{C}^\bullet_G(\bar{\Omega}), d - \bar{\iota} + \alpha + \eta), \qquad \forall * \in \mathbb{Z}/2\mathbb{Z}.$$

Using the above for $A = C_c^\infty(\tilde{H})^{\mathbb{T}}$ and $2\pi i(\alpha + \eta)$ instead of $\alpha + \eta$, and $\Theta = -2\pi i B$, one gets an isomorphism between $HP_G(A)$ and twisted equivariant cohomology of M.

In addition, there is a Chern character map $\mathrm{ch} : K_G(A) \to HP_G(A)$. It is not an isomorphism, but it becomes one after "completing" $K_G(A)$.

To explain what the complex Ω above consists of:

The groupoid $H \rightrightarrows N$ is endowed with a G-invariant pseudo-etale structure, i.e. F is a G-invariant sub-bundle of TH such that $F \rightrightarrows TN$ is a subgroupoid of $TH \rightrightarrows TN$, and such that for all $h \in H$, $s_* : F_h \longrightarrow T_{s(h)}N$ and $r_* : F_h \longrightarrow T_{r(h)}N$ are isomorphisms.

Then Ω is constructed using the convolution algebra

$$\Gamma(H; \wedge^n F^* \otimes L).$$

(Note : if G is an étale groupoid then $\Omega^*(G)$ is a convolution algebra, but if G is not étale one needs a pseudo-étale structure to construct a product.)