Fixed Point Factors under Product Type Actions on ITPFI Factors

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EU-NCG Third Annual Meeting - Cardiff
June 29, 2010

Problem set up

- $A = \otimes_{n \geqslant 1} M_k(\mathbb{C})$ the k$^\infty$-UHF algebra
- Recall: φ state on $M_k(\mathbb{C})$, $\varphi(\cdot) = \mathrm{tr}(h\cdot)$, $h > 0$, $\mathrm{tr}(h) = 1$.
- $\varphi = \otimes_{n \geqslant 1} \varphi_n$ a product state on A where $\varphi_n = \mathrm{tr}\ (h_n \cdot)$, and h_n are diagonal matrices with strictly positive entries and tr $a_n = 1$.
- (H_φ, π_φ) GNS representation of (A, φ).
- $M = \pi_\varphi(A)'' = \otimes_{n \geqslant 1}(M_k(\mathbb{C}), \varphi_n)$ is a factor; such a factor is called Araki-Woods factor or ITPFI factor; ITPFI factors are AFD factors.
- G compact group;
- $\pi : G \to M_k(\mathbb{C})$ unitary representation of G
- $\alpha : G \to$ Aut (A) the product type (Xerox) action induced by the representation π, which is given by

$$\alpha(g) = \otimes_{n \geqslant 1} \mathrm{Ad}\ \pi(g)$$

- A^G the fixed point algebra under α
- fact: A^G is an AF-algebra

Problem set up

- $\varphi^G = \varphi|_{A^G}$
- $(\pi_{\varphi^G}, H_{\varphi^G})$ - the GNS representation of (A^G, φ^G)

Question 1

Find conditions on the product state φ for $\pi_{\varphi^G}(A^G)''$ being a factor and more precisely for being of type II_1, II_∞ or of type III.

We assume first that the action α is induced by a diagonal representation of a compact group G.

In this case the action α induces an action on $M = \pi_\varphi(A)''$, denoted also with α, and

$$M^G \simeq \pi_{\varphi^G}(A^G)''.$$

The 1-Torus and $SU(2)$ action on $\otimes_{n\geqslant 1} M_2(\mathbb{C})$

Let

- $B = \otimes_{n\geqslant 1} M_2(\mathbb{C})$ 2^{∞}-UHF algebra (CAR algebra)
- $\pi : SU(2) \to M_2(\mathbb{C})$, the standrad representation of $SU(2)$, and $\alpha : SU(2) \to \mathrm{Aut}(A)$ the corresponding product action.
- the restriction of α to the maximal torus T is
$$t \mapsto \otimes_{n\geqslant 1} \mathrm{Ad} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}, t \in [0, 2\pi) \,.$$
- $B^{SU(2)}$ and B^{T} - the fixed point algebras
- B^{T} - the GICAR algebra
- $\varphi = \otimes_{n\geqslant 1} \varphi_n$ given by $\varphi_n(\cdot) = \mathrm{tr}(h_n \cdot)$, $h_n = \mathrm{diag}(a_n, b_n)$ $\;a_n, b_n \in (0,1)$ and $a_n + b_n = 1$

The 1-Torus and $SU(2)$ action on $\otimes_{n\geqslant 1} M_2(\mathbb{C})$

Theorem 1

$\pi_{\varphi^T}(B^T)''$ is factor if and only if $\sum\limits_{n=1}^{\infty} a_n(1-a_n)=\infty$.

If $\pi_{\varphi^T}(B^T)''$ is a factor then $\pi_{\varphi^T}(B^T)''$ is

(i) *of type II_1 if and only if there exist $a\in(0,1)$, such that $\sum(a_n-a)^2<\infty$;*

(ii) *of type II_∞ if and only if there exist $a\in(0,1)$, such that $a_n\nrightarrow a$ and $\sum a_n(1-a_n)(a_n-a)^2<\infty$;*

(iii) *of type III if and only if for every $a\in(0,1)$ $\sum a_n(1-a_n)(a_n-a)^2<\infty$.*

If $\sum(a_n-\frac{1}{2})^2=\infty$ then
$\pi_\varphi(B^T)''=\pi_\varphi(B^{SU(2)})''$ and $\pi_{\varphi^T}(B^T)''=\pi_{\varphi^{SU(2)}}(B^{SU(2)})''$

Factoriality of $\pi_{\varphi^G}(A^G)''$

$M = \pi_\varphi(A)'' \simeq M(X, \mu, \mathcal{T})$ and $\pi_{\varphi^G}(A^G)'' \simeq M(X, \mu, \mathcal{R})$, where

- $X = \prod_{n\geqslant 1}\{0, 1, 2, \dots k-1\}$, $\mu = \otimes_{n\geqslant 1}\mu_n$, with $\mu_n(i-1) = \varphi_n(e^n_{ii})$ for $n \geqslant 1$, $1 \leqslant i \leqslant k$ where e^k_{ij}, $i, j \in \{1, 2, \dots, k\}$ are the matrix units of the k^{th} term in the tensor product $\otimes_{n\geqslant 1} M_k(\mathbb{C})$.
- $\mathcal{T}$ is the tail equivalence relation on X, that for $x = (x_n)_{n\geqslant 1}$, $y = (y_n)_{n\geqslant 1}$ is given by

$$x\mathcal{T}y : \exists \ \ n \geqslant 1 \text{ and } x_i = y_i, i > n$$

- $\mathcal{R}$ is a subequivalence relation of $\mathcal{T}$

Factoriality of $\pi_{\varphi^G}(A^G)''$

- $\pi_{\varphi^G}(A^G)''$ factor $\iff \mathcal{R}$ is ergodic with respect to μ.
- Let $\mathcal{R}_\infty$ be the equivalence relation on X induced by the action of S_∞, the group of all finite permutations on $\mathbb{N} \setminus \{0\}$, acting by
$$\sigma(x_1, x_2, \ldots, x_n, \ldots) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}, \ldots),$$
for $x \in X$, $\sigma \in S_\infty$.
- Aldous-Pitman: $\mathcal{R}_\infty$ is ergodic with respect to $\mu = \otimes_{n\geqslant 1}\mu_n$ is and only if
$$\sum \mu_n(B)(1 - \mu_n(B)) = \infty$$
for every $\emptyset \neq B \subseteq \{0, 1, \ldots k-1\}$.
- $\mathcal{R} \supseteq \mathcal{R}_\infty \to$ if $\mathcal{R}_\infty$ ergodic then $\mathcal{R}$ ergodic

Example 2

For the GICAR case, $\pi_{\varphi^T}(B^T)'' \simeq M(X, \mu, \mathcal{R}_\infty)$, where

- $X = \prod_{n\geqslant 1}\{0, 1\}$
- $\mu = \otimes_{n\geqslant 1}\mu_n$, where $\mu_n(0) = a_n$, $\mu_n(1) = b_n$

Factoriality of $\pi_{\varphi^G}(A^G)''$

Example 3

- $A = \otimes_{n\geqslant 1} M_3(\mathbb{C})$ the 3^∞-UHF algebra
- $\alpha : T \to \text{Aut }(A)$, $\alpha(t) = \otimes_{n\geqslant 1}\text{Ad}\begin{bmatrix} e^{it} & & \\ & 1 & \\ & & e^{-it} \end{bmatrix}$
- $\varphi = \otimes_{n\geqslant 1}\varphi_n,\ \varphi_n(\cdot) = \text{tr}(h_n\cdot),\ h_n = \text{diag}(a_n, b_n, c_n) > 0,\ \text{tr}(h_n) = 1.$

Then $N = M^T \simeq \pi_{\varphi^T}(A^T)'' \simeq M(X, \mu, \mathcal{R})$, where

- $X = \prod_{n\geqslant 1}\{0,1,2\}$,
- $\mu_n(0) = a_n,\ \mu_n(1) = b_n,\ \mu_n(2) = c_n,\ n \geqslant 1$
- $x\mathcal{R}y \Leftrightarrow \exists n,\ x_i = y_i,\ i > n$ and $\sum\limits_{i=1}^{n}(x_i - y_i) = 0$

Proposition 4

$N \simeq M(X, \mu, \mathcal{R})$ is factor if and only if $\sum b_n(1 - b_n) = \infty$.

Fixed point factors of type II_1

$\pi_{\varphi^G}(A^G)''$ is of type $\mathrm{II}_1 \Leftrightarrow \mu$ is equivalent to an ergodic $\mathcal{R}$-invariant probability measure on X.

Non-atomic fully supported $\mathcal{R}$-invariant ergodic measures on $X \longleftrightarrow$ Faithful extremal traces of the fixed point algebra A^G

Handelman: For the above example, the faithful diagonal extremal traces are of the form $\otimes_{n\geqslant 1}\mathrm{tr}(h\cdot)$, where

$$h = \mathrm{diag}\left(\frac{x}{1+x+\frac{1}{x}}, \frac{1}{1+x+\frac{1}{x}}, \frac{\frac{1}{x}}{1+x+\frac{1}{x}}\right), \ x > 0$$

Proposition 5

$N \simeq \pi_{\varphi^T}(A^T)''$ is of type II_1 if and only if there exists $x > 0$, such that

$$\sum (a_n - a)^2 + (b_n - b)^2 + (c_n - c)^2 < \infty,$$

where $a = \frac{x}{1+x+\frac{1}{x}}$, $b = \frac{1}{1+x+\frac{1}{x}}$, $c = \frac{\frac{1}{x}}{1+x+\frac{1}{x}}$.

Fixed point factors of type II_∞ and of type III

- Connes → invariant T
- N, $\varphi \to \sigma_t^\varphi$, the modular group of automorphisms
- $T(N) = \{t \in \mathbb{R}, \sigma_t^\varphi \text{ inner}\}$
- If a factor N has separable predual, then

$$N \text{ is semifinite } \Leftrightarrow T(N) = \mathbb{R}$$

Proposition 6

If M^G is a fixed point factor under a product type action α of G on $M = \pi_\varphi(A)''$, then

$$t \in T(M^G) \Leftrightarrow \exists\ v \in U(M^G) \text{ and } g \in G \text{ such that } \sigma_t^\varphi = Ad\ v \circ \alpha_g$$

where σ_t^φ is the modular group of automorphisms of M.

Fixed point factors of type II_∞ and of type III

For the example considered, we have that:

Proposition 7

$t \in T(N)$ if and only if there exist $s \in [0, 2\pi)$ such that

$$\sum \Big[a_n b_n (1 - \cos(t \log \frac{a_n}{b_n} - s)) + b_n c_n (1 - \cos(t \log \frac{b_n}{c_n} - s)) + a_n c_n (1 - \cos(t \log \frac{c_n}{a_n} + 2s)) \Big] < \infty$$

Fixed point factors of type II_∞ and of type III

Theorem 8 (Giordano-M.)

(1) If $\sum a_n(1-a_n)=\infty$, $\sum b_n(1-b_n)=\infty$, $\sum c_n(1-c_n)=\infty$, *then* $N \simeq \pi_{\varphi^T}(A^T)''$ *is a factor, and*

(i) N *is of type* II_1 *if and only if there exists* $x>0$ *such that:*

$$\sum (a_n-a)^2+(c_n-c)^2+(b_n-b)^2<\infty,$$

where $a=\frac{x}{1+x+\frac{1}{x}}$, $b=\frac{1}{1+x+\frac{1}{x}}$, *and* $c=\frac{\frac{1}{x}}{1+x+\frac{1}{x}}$.

If N *is not of type* II_1 *then:*

(ii) N *is of type* II_∞ *if and only if there exists* $x>0$ *such that:*

$$\sum a_nb_n(a_nb-b_na)^2+a_nc_n(a_nc-c_na)^2+b_nc_n(c_nb-b_nc)^2<\infty$$

where $a=\frac{x}{1+x+\frac{1}{x}}$, $b=\frac{1}{1+x+\frac{1}{x}}$, *and* $c=\frac{\frac{1}{x}}{1+x+\frac{1}{x}}$;

(iii) N *is a factor of type III otherwise.*

Fixed point factors of type II_∞ and of type III

(2) If $\sum b_n(1-b_n)=\infty$ *and* $\sum a_n(1-a_n)<\infty$*, then,* N *is of type* II_∞ *if and only if*

$$\sum b_n c_n \left(b_n c - c_n b\right)^2 < \infty$$

for a unique pair (a,c) *with* $b+c=1$*. Otherwise,* N *is of type III.*

(3) If $\sum b_n(1-b_n)=\infty$ *and* $\sum c_n(1-c_n)<\infty$*, then,* N *is of type* II_∞ *if and only if*

$$\sum a_n b_n \left(a_n b - b_n a\right)^2 < \infty,$$

for a unique pair (a,b) *with* $a+b=1$*. Otherwise,* N *is of type III.*

Subtypes for type III factors

Recall

- Connes → invariant S for factors
- $0 \in S(M)$ if and only if M type III.
- $S(M) \cap \mathbb{R}_+^*$ closed subgroup of $\mathbb{R}_+^*$

If M is a factor of type III

- $S(M) = \{0, 1\} \longrightarrow III_0$,
- $S(M) = \{0\} \cup \{\lambda^n : n \in Z\}$ for $\lambda \in (0, 1) \longrightarrow III_\lambda$,
- $S(M) = [0, \infty) \longrightarrow III_1$.
- for ITPFI factors, Connes' invariant S corresponds to asymptotic ratio set defined by Araki-Woods

We constructed a ratio set, $r_\infty(N,\varphi)$, for factors of the form $N = \pi_\varphi(A^G)''$ which are fixed point factors under diagonal product type actions, where α is a product action induced by a diagonal representation of G on A.

Theorem 9 (Giordano-M.)

If N is of type III then:

(i) *N is of type III_0 if and only if $r_\infty(N,\varphi) \setminus \{0\} = \{1\}$;*

(ii) *N is of type III_λ for some $\lambda \in (0,1)$ if and only if $r_\infty(N,\varphi) \setminus \{0\} = \{\lambda^n,\ n \in \mathbb{Z}\}$;*

(iii) *N is of type III_1 if and only if $r_\infty(N,\varphi) \setminus \{0\} = (0,\infty)$.*

Fixed Point which are ITPFI Factors

Question 2

Are fixed point factors are ITPFI factors?

Recall

- Fixed point factors are AFD factors
- There is a unique AFD factor of type II_1, II_∞, III_λ with $\lambda \neq 0$
- In 1980, Connes and Woods introduced a new property of ergodic actions called approximate transitivity to characterize among AFD von Neumann factors the Araki-Woods or ITPFI factors.

Fixed Point Factors which are ITPFI Factors

Definition 10 (Approximate Transitivity)

An action of a group G on a measure space (X, μ) is *approximately transitive (AT)* if for all $\epsilon > 0$ and any sequence f_1, f_2, . . . , f_n of functions in $L^1(X, \mu)_+$, the space of positive integrable functions, there exists a function f in $L^1(X, \mu)_+$, finitely many elements $g_{i,j}$ of G and constants $\lambda_{i,j} \geqslant 0$ such that

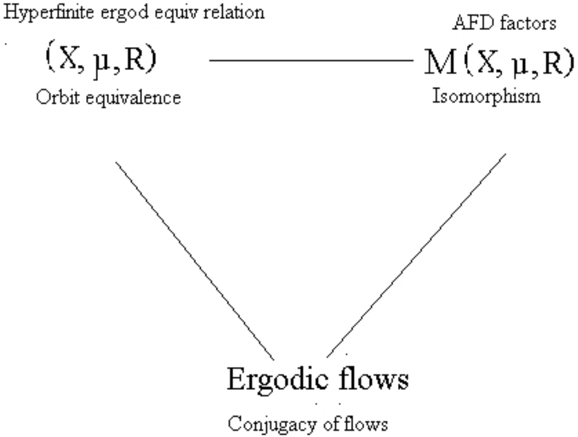
$$\left\| f_i - \sum_j \lambda_{i,j} f \circ g_{i,j} \frac{d\mu \circ g_{i,j}}{d\mu} \right\|_1 < \epsilon$$

for each i.

Definition 11 (Associated Flow)

Let $\mathcal{R}$ be an equivalence relation on $(X, \mathfrak{B}, \mu)$. Let $\widetilde{\mathcal{R}}$ be the the equivalence relation on $(X \times \mathbb{R}, \mu \times e^u du)$ defined by $((x, s), (y, t)) \in \widetilde{\mathcal{R}}$ if $(x, y) \in \mathcal{R}$ and $t = s - \log \delta(y, x)$, where $\delta(y, x)$ is the Radon-Nicodym cocycle. By $\mathcal{F}$ we denote the sub σ-algebra consisting of all $\widetilde{\mathcal{R}}$-invariant sets. By Y we denote the quotient space $X \times \mathbb{R}/\mathcal{F}$, that is the space of all $\widetilde{\mathcal{R}}$-ergodic components. We let π be the natural surjection from $X \times \mathbb{R}$ to Y. By $\{T_t, \ t \in \mathbb{R}\}$, we denote the flow $T_t(x, s) = (x, s + t)$ for $(x, s) \in X \times \mathbb{R}$. By $\{F_t, \ t \in \mathbb{R}\}$, we denote the factor flow of $\{T_t, \ t \in \mathbb{R}\}$ to the quotient space Y through the factor map π, that is, $\pi T_t = F_t \pi$, for all $t \in \mathbb{R}$. The flow F_t is called the *associated flow* (or the *Poincaré flow*) of $\mathcal{R}$.

Fixed Point Factors which are ITPFI Factors



Hyperfinite ergod equiv relation

(X, μ, R)

Orbit equivalence

AFD factors

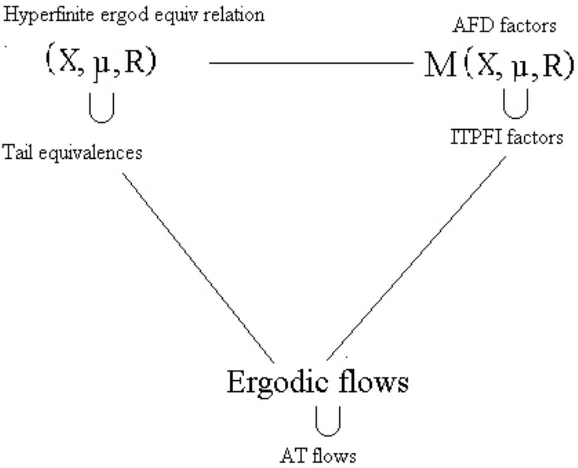
$M(X, \mu, R)$

Isomorphism

Ergodic flows

Conjugacy of flows

Fixed Point Factors which are ITPFI Factors



Connes-Woods: An AFD factor is ITPFI if and only if its flow of weights is AT.

Fixed Point Factors which are ITPFI Factors

We recall the following results:

- Giordano-Handelman: fixed point factors under Z_2-actions on ITPFI$_2$ are ITPFI factors,
- Giordano-Handelman: constructed a fixed point factors under a product action of Z_2 on an unbounded ITPFI factor - $M = \otimes(M_{k_n}(\mathbb{C}), \varphi_n)$, $\sup k_n = \infty$ - which is not ITPFI factor.
  - It is the Krieger factor of a non-singular transformation T that is very "close" to a odometer, but it is not orbit equivalent to any product odometer, since its associated flow is not AT.
  - Recall: Krieger introduced the so called property A, and he proved that any product type odometer satisfies this property.

Theorem 12 (M.)

The above transformation T, satisfies Krieger's property A

This result answers a question of Dooley and Hamachi, namely, wether property A implies AT or not.

Fixed Point Factors which are ITPFI Factors

Theorem 13 (Giordano-M.)

1. *if* M^G *is the fixed point factor under a product type action on* $\otimes(M_k, \varphi_n)$ *and the sequence* $(h_n)_{n\geqslant 1}$ *of density matrices of* φ_n *has a limit point with non zero entries,* M^G *is an ITPFI factor;*
2. $M = \otimes(M_2(\mathbb{C}), tr(h_i\cdot))$ *with*

$$h_i = diag(\frac{1}{1+\lambda_n}, \frac{\lambda_n}{1+\lambda_n})$$

for $l_0 + l_1 + l_2 + \cdots + l_{n-1} < i \leqslant l_0 + l_1 + l_2 + \cdots + l_{n-1} + l_n$, $l_0 = 0$, $\lambda_n \to 0$ *and* $\lambda_n l_n \to \infty$ *then* M^T *is an ITPFI factor.*

3. *there exists fixed point factors under actions of finite groups on unbounded ITPFI factors -* $\otimes_{n\geqslant 1}(M_{k_n}(\mathbb{C}), tr(h_n\cdot))$, $\sup k_n = \infty$ *- that are ITPFI.*
4. *Any Araki-Woods factor of the form* $\otimes_{n\geqslant 1}(M_k(\mathbb{C}), tr(h_n\cdot))$ *can obtained as a fixed point factor.*

Non-diagonal Product Actions

Recall

Question 1

Find conditions on the product state φ for $\pi_{\varphi^G}(A^G)''$ being a factor and more precisely for being of type II_1, II_∞ or of type *III*.

Question 3

How do we answer Question 1 if α is a product type action induced by non-diagonal representations of compact groups?

Non-diagonal Product Actions

Answer: By reducing it to the study of a diagonal Xerox action.

- $\varphi = \otimes_{n\geqslant 1}\mathrm{tr}(h_n\cdot)$ be a faithful diagonal product state on $A = \otimes_{n\geqslant 1} M_k(\mathbb{C})$
- $\alpha(g) = \otimes_{n\geqslant 1}\mathrm{Ad}\ \pi(g)$ with π a representation (non-diagonal) of G on $\mathcal{U}(M_k(\mathbb{C}))$
- $G_0 = \{g \in G, \pi(g) \text{ diagonal matrix}\}$
- $\varphi^{G_0} = \varphi|_{A^{G_0}}$
- $(h_n)_{n\geqslant 1}$ has an accumulation point h, with mutual distinct entries on the diagonal.

With the above conditions, we proved that

$$\pi_\varphi(A^G)'' = \pi_\varphi(A^{G_0})''$$

and

$$\pi_{\varphi^G}(A^G)'' \simeq \pi_{\varphi^{G_0}}(A^{G_0})''$$

Non-diagonal Product Actions

On $A = \otimes_{n \geqslant 1} M_3(\mathbb{C})$ we consider the product state $\varphi = \otimes_{n \geqslant 1} \varphi_n$
$\varphi_n(\cdot) = \mathrm{tr}(h_n \cdot)$ and $h_n = \mathrm{diag}(a_n, b_n, c_n)$, $a_n, b_n, c_n > 0$ $a_n + b_n + c_n = 1$.

(1) Let $\pi : SU(2) \to M_3(\mathbb{C})$ given by

$$\pi \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} = \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ -\overline{b} & 0 & \overline{a} \end{bmatrix}$$

$A^{SU(2)}$-the fied point algebra.

$T \subseteq SU(2)$

$$\pi|_T(t) = \begin{bmatrix} e^{it} & & \\ & 1 & \\ & & e^{-it} \end{bmatrix}$$

If (a_n, b_n, c_n) has a limit point (a, b, c) with $a \neq c$, then

$$\pi_\varphi(A^T)'' = \pi_\varphi(A^{SU(2)})''$$

Non-diagonal Product Actions

(2) $A^{SU(3)}$ - the fixed point algebra under the standard Xerox type action of $SU(3)$ on $A = \otimes_{n \geqslant 1} M_3(\mathbb{C})$

A^{T^2} - the fixed point algebra under the restriction of this action to T^2, the maximal torus of $SU(3)$.

If (a_n, b_n, c_n) has a limit point (a, b, c), with a, b and c mutually distinct, then

$$\pi_\varphi(A^{T^2})'' = \pi_\varphi(A^{SU(3)})''$$