

# Symmetries and invariance principles in noncommutative probability

Claus Köstler

Institute of Mathematics and Physics  
Aberystwyth University

`cck@aber.ac.uk`

(with Rolf Gohm and Roland Speicher)

*EU-NCG Third Annual Meeting*  
School of Mathematics  
Cardiff University

June 28, 2010

# Motivation

*Though many probabilistic symmetries are conceivable [...], four of them - **stationarity**, **contractability**, **exchangeability** and **rotatability** - stand out as especially interesting and important in several ways: Their study leads to some **deep structural theorems** of great beauty and significance [...].*

*Olav Kallenberg (2005)*

# Motivation

*Though many probabilistic symmetries are conceivable [...], four of them - **stationarity**, **contractability**, **exchangeability** and **rotatability** - stand out as especially interesting and important in several ways: Their study leads to some **deep structural theorems** of great beauty and significance [...].*

*Olav Kallenberg (2005)*

## Quest

Transfer the related concepts to **noncommutative probability** and show that they turn out to be fruitful in the study of the structure of **operator algebras** and **quantum dynamics**.

# Motivation

*Though many probabilistic symmetries are conceivable [...], four of them - **stationarity**, **contractability**, **exchangeability** and **rotatability** - stand out as especially interesting and important in several ways: Their study leads to some **deep structural theorems** of great beauty and significance [...].*

*Olav Kallenberg (2005)*

## Quest

Transfer the related concepts to **noncommutative probability** and show that they turn out to be fruitful in the study of the structure of **operator algebras** and **quantum dynamics**.

## Remark

Noncommutative probability = classical & quantum probability

# Foundational result on distributional symmetries and invariance principles in classical probability

The random variables  $(X_n)_{n \in \mathbb{N}}$  are said to be **exchangeable** if

$$\mathbb{E}(X_{\mathbf{i}(1)} \cdots X_{\mathbf{i}(n)}) = \mathbb{E}(X_{\sigma(\mathbf{i}(1))} \cdots X_{\sigma(\mathbf{i}(n))}) \quad (\sigma \in \mathbb{S}_\infty)$$

for all  $n$ -tuples  $\mathbf{i}: \{1, 2, \dots, n\} \rightarrow \mathbb{N}$  and  $n \in \mathbb{N}$ .

# Foundational result on distributional symmetries and invariance principles in classical probability

The random variables  $(X_n)_{n \in \mathbb{N}}$  are said to be **exchangeable** if

$$\mathbb{E}(X_{\mathbf{i}(1)} \cdots X_{\mathbf{i}(n)}) = \mathbb{E}(X_{\sigma(\mathbf{i}(1))} \cdots X_{\sigma(\mathbf{i}(n))}) \quad (\sigma \in \mathbb{S}_\infty)$$

for all  $n$ -tuples  $\mathbf{i}: \{1, 2, \dots, n\} \rightarrow \mathbb{N}$  and  $n \in \mathbb{N}$ .

**Theorem (De Finetti 1931, ...)**

The law of an exchangeable sequence  $(X_n)_{n \in \mathbb{N}_0}$  is given by a unique convex combination of infinite product measures. More intuitively:

*"Any exchangeable process is an average of i.i.d. processes."*

# Foundational result on the infinite symmetric group $\mathbb{S}_\infty$

$\mathbb{S}_\infty$  is the inductive limit of the symmetric group  $\mathbb{S}_n$  as  $n \rightarrow \infty$ . A function  $\chi: \mathbb{S}_\infty \rightarrow \mathbb{C}$  is a **character** if it is constant on conjugacy classes, positive definite and normalized at the unity.

# Foundational result on the infinite symmetric group $\mathbb{S}_\infty$

$\mathbb{S}_\infty$  is the inductive limit of the symmetric group  $\mathbb{S}_n$  as  $n \rightarrow \infty$ . A function  $\chi: \mathbb{S}_\infty \rightarrow \mathbb{C}$  is a **character** if it is constant on conjugacy classes, positive definite and normalized at the unity.

## Theorem (Thoma 1964)

An extremal character of the group  $\mathbb{S}_\infty$  is of the form

$$\chi(\sigma) = \prod_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} a_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} b_j^k \right)^{m_k(\sigma)}.$$

Here  $m_k(\sigma)$  is the number of  $k$ -cycles in the permutation  $\sigma$  and the two sequences  $(a_i)_{i=1}^{\infty}, (b_j)_{j=1}^{\infty}$  satisfy

$$a_1 \geq a_2 \geq \dots \geq 0, \quad b_1 \geq b_2 \geq \dots \geq 0, \quad \sum_{i=1}^{\infty} a_i + \sum_{j=1}^{\infty} b_j \leq 1.$$

# Thoma's theorem as a noncommutative de Finetti theorem

## Alternative proofs

Kerov & Vershik 1981: asymptotic representation theory

Okounkov 1997: Olshanski semigroups and spectral theory

# Thoma's theorem as a noncommutative de Finetti theorem

## Alternative proofs

Kerov & Vershik 1981: asymptotic representation theory

Okounkov 1997: Olshanski semigroups and spectral theory

## Key fact of our new operator algebraic proof

Let  $\gamma_i := (0, i)$ . Then the sequence  $(\gamma_i)_{i \in \mathbb{N}}$  is 'exchangeable', i.e.

$$\chi(\gamma_{\mathbf{i}(1)} \gamma_{\mathbf{i}(2)} \cdots \gamma_{\mathbf{i}(n)}) = \chi(\gamma_{\sigma(\mathbf{i}(1))} \gamma_{\sigma(\mathbf{i}(2))} \cdots \gamma_{\sigma(\mathbf{i}(n))}) \quad (\sigma \in \mathbb{S}_\infty)$$

for all  $n$ -tuples  $\mathbf{i}: \{1, \dots, n\} \rightarrow \mathbb{N}$  and  $n \in \mathbb{N}$ .

# Thoma's theorem as a noncommutative de Finetti theorem

## Alternative proofs

Kerov & Vershik 1981: asymptotic representation theory

Okounkov 1997: Olshanski semigroups and spectral theory

## Key fact of our new operator algebraic proof

Let  $\gamma_i := (0, i)$ . Then the sequence  $(\gamma_i)_{i \in \mathbb{N}}$  is 'exchangeable', i.e.

$$\chi(\gamma_{i(1)}\gamma_{i(2)} \cdots \gamma_{i(n)}) = \chi(\gamma_{\sigma(i(1))}\gamma_{\sigma(i(2))} \cdots \gamma_{\sigma(i(n))}) \quad (\sigma \in \mathbb{S}_\infty)$$

for all  $n$ -tuples  $\mathbf{i}: \{1, \dots, n\} \rightarrow \mathbb{N}$  and  $n \in \mathbb{N}$ .

## Task

Identify the convex combination of extremal characters of  $\mathbb{S}_\infty$ . In other words: prove a noncommutative de Finetti theorem!

# Thoma's theorem as a noncommutative de Finetti theorem

## Alternative proofs

Kerov & Vershik 1981: asymptotic representation theory

Okounkov 1997: Olshanski semigroups and spectral theory

## Key fact of our new operator algebraic proof

Let  $\gamma_i := (0, i)$ . Then the sequence  $(\gamma_i)_{i \in \mathbb{N}}$  is 'exchangeable', i.e.

$$\chi(\gamma_{i(1)}\gamma_{i(2)} \cdots \gamma_{i(n)}) = \chi(\gamma_{\sigma(i(1))}\gamma_{\sigma(i(2))} \cdots \gamma_{\sigma(i(n))}) \quad (\sigma \in \mathbb{S}_\infty)$$

for all  $n$ -tuples  $\mathbf{i}: \{1, \dots, n\} \rightarrow \mathbb{N}$  and  $n \in \mathbb{N}$ .

## Task

Identify the convex combination of extremal characters of  $\mathbb{S}_\infty$ . In other words: prove a noncommutative de Finetti theorem!

## Reference:

R. Gohm & C. Köstler. *Noncommutative independence from characters of the symmetric group*  $\mathbb{S}_\infty$ . 47 pages. Preprint (2010). ([arXiv:1005.5726](https://arxiv.org/abs/1005.5726))

# Noncommutative distributions

A (noncommutative) **probability space**  $(\mathcal{A}, \varphi)$  is a von Neumann algebra  $\mathcal{A}$  (with separable predual) equipped with a faithful normal state  $\varphi$ .

# Noncommutative distributions

A (noncommutative) **probability space**  $(\mathcal{A}, \varphi)$  is a von Neumann algebra  $\mathcal{A}$  (with separable predual) equipped with a faithful normal state  $\varphi$ .

Two sequences  $(x_n)_{n \geq 0}$  and  $(y_n)_{n \geq 0}$  in  $\mathcal{A}$  have the same **distribution** if

$$\varphi(x_{\mathbf{i}(1)} x_{\mathbf{i}(2)} \cdots x_{\mathbf{i}(n)}) = \varphi(y_{\mathbf{i}(1)} y_{\mathbf{i}(2)} \cdots y_{\mathbf{i}(n)})$$

for all  $n$ -tuples  $\mathbf{i}: \{1, 2, \dots, n\} \rightarrow \mathbb{N}_0$  and  $n \in \mathbb{N}$ .

# Noncommutative distributions

A (noncommutative) **probability space**  $(\mathcal{A}, \varphi)$  is a von Neumann algebra  $\mathcal{A}$  (with separable predual) equipped with a faithful normal state  $\varphi$ .

Two sequences  $(x_n)_{n \geq 0}$  and  $(y_n)_{n \geq 0}$  in  $\mathcal{A}$  have the same **distribution** if

$$\varphi(x_{i(1)} x_{i(2)} \cdots x_{i(n)}) = \varphi(y_{i(1)} y_{i(2)} \cdots y_{i(n)})$$

for all  $n$ -tuples  $\mathbf{i}: \{1, 2, \dots, n\} \rightarrow \mathbb{N}_0$  and  $n \in \mathbb{N}$ .

**Notation**

$$(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (y_0, y_1, y_2, \dots)$$

# Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A sequence  $(x_n)_{n \geq 0}$  is

- **exchangeable** if  $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_{\pi(0)}, x_{\pi(1)}, x_{\pi(2)}, \dots)$  for any (finite) permutation  $\pi \in \mathbb{S}_\infty$  of  $\mathbb{N}_0$ .

# Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A sequence  $(x_n)_{n \geq 0}$  is

- **exchangeable** if  $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_{\pi(0)}, x_{\pi(1)}, x_{\pi(2)}, \dots)$  for any (finite) permutation  $\pi \in \mathbb{S}_\infty$  of  $\mathbb{N}_0$ .
- **spreadable** if  $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_{n_0}, x_{n_1}, x_{n_2}, \dots)$  for any subsequence  $(n_0, n_1, n_2, \dots)$  of  $(0, 1, 2, \dots)$ .

# Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A sequence  $(x_n)_{n \geq 0}$  is

- **exchangeable** if  $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_{\pi(0)}, x_{\pi(1)}, x_{\pi(2)}, \dots)$  for any (finite) permutation  $\pi \in \mathbb{S}_\infty$  of  $\mathbb{N}_0$ .
- **spreadable** if  $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_{n_0}, x_{n_1}, x_{n_2}, \dots)$  for any subsequence  $(n_0, n_1, n_2, \dots)$  of  $(0, 1, 2, \dots)$ .
- **stationary** if  $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_k, x_{k+1}, x_{k+2}, \dots)$  for all  $k \in \mathbb{N}$ .

# Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A sequence  $(x_n)_{n \geq 0}$  is

- **exchangeable** if  $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_{\pi(0)}, x_{\pi(1)}, x_{\pi(2)}, \dots)$  for any (finite) permutation  $\pi \in \mathbb{S}_\infty$  of  $\mathbb{N}_0$ .
- **spreadable** if  $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_{n_0}, x_{n_1}, x_{n_2}, \dots)$  for any subsequence  $(n_0, n_1, n_2, \dots)$  of  $(0, 1, 2, \dots)$ .
- **stationary** if  $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_k, x_{k+1}, x_{k+2}, \dots)$  for all  $k \in \mathbb{N}$ .
- **identically distributed** if  $(x_k, x_k, x_k, \dots) \stackrel{\text{distr}}{=} (x_l, x_l, x_l, \dots)$  for all  $k, l \in \mathbb{N}_0$

# Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A sequence  $(x_n)_{n \geq 0}$  is

- **exchangeable** if  $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_{\pi(0)}, x_{\pi(1)}, x_{\pi(2)}, \dots)$  for any (finite) permutation  $\pi \in \mathbb{S}_\infty$  of  $\mathbb{N}_0$ .
- **spreadable** if  $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_{n_0}, x_{n_1}, x_{n_2}, \dots)$  for any subsequence  $(n_0, n_1, n_2, \dots)$  of  $(0, 1, 2, \dots)$ .
- **stationary** if  $(x_0, x_1, x_2, \dots) \stackrel{\text{distr}}{=} (x_k, x_{k+1}, x_{k+2}, \dots)$  for all  $k \in \mathbb{N}$ .
- **identically distributed** if  $(x_k, x_k, x_k, \dots) \stackrel{\text{distr}}{=} (x_l, x_l, x_l, \dots)$  for all  $k, l \in \mathbb{N}_0$

## Lemma (Hierarchy of distributional symmetries)

exchangeability  $\Rightarrow$  spreadability  $\Rightarrow$  stationarity  $\Rightarrow$  identical distr.

# Noncommutative conditional independence

Let  $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$  be three von Neumann subalgebras of  $\mathcal{A}$  with  $\varphi$ -preserving conditional expectations  $E_i: \mathcal{A} \rightarrow \mathcal{A}_i$  ( $i = 0, 1, 2$ ).

# Noncommutative conditional independence

Let  $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$  be three von Neumann subalgebras of  $\mathcal{A}$  with  $\varphi$ -preserving conditional expectations  $E_i: \mathcal{A} \rightarrow \mathcal{A}_i$  ( $i = 0, 1, 2$ ).

## Definition

$\mathcal{A}_1$  and  $\mathcal{A}_2$  are  $\mathcal{A}_0$ -**independent** if

$$E_0(xy) = E_0(x)E_0(y) \quad (x \in \mathcal{A}_0 \vee \mathcal{A}_1, y \in \mathcal{A}_0 \vee \mathcal{A}_2)$$

# Noncommutative conditional independence

Let  $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$  be three von Neumann subalgebras of  $\mathcal{A}$  with  $\varphi$ -preserving conditional expectations  $E_i: \mathcal{A} \rightarrow \mathcal{A}_i$  ( $i = 0, 1, 2$ ).

## Definition

$\mathcal{A}_1$  and  $\mathcal{A}_2$  are  $\mathcal{A}_0$ -**independent** if

$$E_0(xy) = E_0(x)E_0(y) \quad (x \in \mathcal{A}_0 \vee \mathcal{A}_1, y \in \mathcal{A}_0 \vee \mathcal{A}_2)$$

## Remarks

- $\mathcal{A}_0 \subset \mathcal{A}_0 \vee \mathcal{A}_1, \mathcal{A}_0 \vee \mathcal{A}_2 \subset \mathcal{A}$  is a **commuting square**
- If  $\mathcal{A} = L^\infty(\Omega, \Sigma, \mu)$  we obtain conditional independence with respect to a sub- $\sigma$ -algebra.

# Noncommutative conditional independence

Let  $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$  be three von Neumann subalgebras of  $\mathcal{A}$  with  $\varphi$ -preserving conditional expectations  $E_i: \mathcal{A} \rightarrow \mathcal{A}_i$  ( $i = 0, 1, 2$ ).

## Definition

$\mathcal{A}_1$  and  $\mathcal{A}_2$  are  $\mathcal{A}_0$ -**independent** if

$$E_0(xy) = E_0(x)E_0(y) \quad (x \in \mathcal{A}_0 \vee \mathcal{A}_1, y \in \mathcal{A}_0 \vee \mathcal{A}_2)$$

## Remarks

- $\mathcal{A}_0 \subset \mathcal{A}_0 \vee \mathcal{A}_1, \mathcal{A}_0 \vee \mathcal{A}_2 \subset \mathcal{A}$  is a **commuting square**
- If  $\mathcal{A} = L^\infty(\Omega, \Sigma, \mu)$  we obtain conditional independence with respect to a sub- $\sigma$ -algebra.
- Many different forms of noncommutative independence!

# Noncommutative conditional independence

Let  $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$  be three von Neumann subalgebras of  $\mathcal{A}$  with  $\varphi$ -preserving conditional expectations  $E_i: \mathcal{A} \rightarrow \mathcal{A}_i$  ( $i = 0, 1, 2$ ).

## Definition

$\mathcal{A}_1$  and  $\mathcal{A}_2$  are  $\mathcal{A}_0$ -**independent** if

$$E_0(xy) = E_0(x)E_0(y) \quad (x \in \mathcal{A}_0 \vee \mathcal{A}_1, y \in \mathcal{A}_0 \vee \mathcal{A}_2)$$

## Remarks

- $\mathcal{A}_0 \subset \mathcal{A}_0 \vee \mathcal{A}_1, \mathcal{A}_0 \vee \mathcal{A}_2 \subset \mathcal{A}$  is a **commuting square**
- If  $\mathcal{A} = L^\infty(\Omega, \Sigma, \mu)$  we obtain conditional independence with respect to a sub- $\sigma$ -algebra.
- Many different forms of noncommutative independence!
- $\mathbb{C}$ -independence & Speicher's universality rules  
 $\rightsquigarrow$  tensor independence or free independence

# Conditional independence of sequences

A sequence  $(x_n)_{n \in \mathbb{N}_0}$  in  $\mathcal{A}$  is **(full)  $\mathcal{B}$ -independent** if

$$\bigvee \{x_i \mid i \in I\} \vee \mathcal{B} \quad \text{and} \quad \bigvee \{x_j \mid j \in J\} \vee \mathcal{B}$$

are  $\mathcal{B}$ -independent whenever  $I \cap J = \emptyset$  with  $I, J \subset \mathbb{N}_0$ .

# Noncommutative extended De Finetti theorem

Let  $(x_n)_{n \in \mathbb{N}_0}$  be random variables in  $(\mathcal{A}, \varphi)$  with tail algebra

$$\mathcal{A}^{\text{tail}} := \bigcap_{n \geq 0} \bigvee_{k \geq n} \{x_k\},$$

and consider:

- (a)  $(x_n)_{n \in \mathbb{N}_0}$  is exchangeable
- (c)  $(x_n)_{n \in \mathbb{N}_0}$  is spreadable
- (d)  $(x_n)_{n \in \mathbb{N}_0}$  is stationary and  $\mathcal{A}^{\text{tail}}$ -independent
- (e)  $(x_n)_{n \in \mathbb{N}_0}$  is identically distributed and  $\mathcal{A}^{\text{tail}}$ -independent

# Noncommutative extended De Finetti theorem

Let  $(x_n)_{n \in \mathbb{N}_0}$  be random variables in  $(\mathcal{A}, \varphi)$  with tail algebra

$$\mathcal{A}^{\text{tail}} := \bigcap_{n \geq 0} \bigvee_{k \geq n} \{x_k\},$$

and consider:

- (a)  $(x_n)_{n \in \mathbb{N}_0}$  is exchangeable
- (c)  $(x_n)_{n \in \mathbb{N}_0}$  is spreadable
- (d)  $(x_n)_{n \in \mathbb{N}_0}$  is stationary and  $\mathcal{A}^{\text{tail}}$ -independent
- (e)  $(x_n)_{n \in \mathbb{N}_0}$  is identically distributed and  $\mathcal{A}^{\text{tail}}$ -independent

Theorem (K. '07-'08 )

(a)  $\Rightarrow$  (c)  $\Rightarrow$  (d)  $\Rightarrow$  (e),

# Noncommutative extended De Finetti theorem

Let  $(x_n)_{n \in \mathbb{N}_0}$  be random variables in  $(\mathcal{A}, \varphi)$  with tail algebra

$$\mathcal{A}^{\text{tail}} := \bigcap_{n \geq 0} \bigvee_{k \geq n} \{x_k\},$$

and consider:

- (a)  $(x_n)_{n \in \mathbb{N}_0}$  is exchangeable
- (c)  $(x_n)_{n \in \mathbb{N}_0}$  is spreadable
- (d)  $(x_n)_{n \in \mathbb{N}_0}$  is stationary and  $\mathcal{A}^{\text{tail}}$ -independent
- (e)  $(x_n)_{n \in \mathbb{N}_0}$  is identically distributed and  $\mathcal{A}^{\text{tail}}$ -independent

Theorem (K. '07-'08, Gohm & K. '08)

$(a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$ , but  $(a) \not\Leftarrow (c) \not\Leftarrow (d) \not\Leftarrow (e)$

# Noncommutative extended De Finetti theorem

Let  $(x_n)_{n \in \mathbb{N}_0}$  be random variables in  $(\mathcal{A}, \varphi)$  with tail algebra

$$\mathcal{A}^{\text{tail}} := \bigcap_{n \geq 0} \bigvee_{k \geq n} \{x_k\},$$

and consider:

- (a)  $(x_n)_{n \in \mathbb{N}_0}$  is exchangeable
- (c)  $(x_n)_{n \in \mathbb{N}_0}$  is spreadable
- (d)  $(x_n)_{n \in \mathbb{N}_0}$  is stationary and  $\mathcal{A}^{\text{tail}}$ -independent
- (e)  $(x_n)_{n \in \mathbb{N}_0}$  is identically distributed and  $\mathcal{A}^{\text{tail}}$ -independent

Theorem (K. '07-'08, Gohm & K. '08)

$(a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$ , but  $(a) \not\Leftarrow (c) \not\Leftarrow (d) \not\Leftarrow (e)$

Remark

(a) to (e) are equivalent if the operators  $x_n$  mutually commute.  
[De Finetti '31, Ryll-Nardzewski '57, ..., Størmer '69, ... Hudson '76, ... Petz '89, ...]

## Intermediate discussion

- Noncommutative conditional independence emerges from distributional symmetries in terms of commuting squares

For further details see:

C. Köstler. *A noncommutative extended de Finetti theorem*. *J. Funct. Anal.* **258**, 1073-1120 (2010)

# Intermediate discussion

- Noncommutative conditional independence emerges from distributional symmetries in terms of commuting squares
- Exchangeability is too weak to identify the structure of the underlying noncommutative probability space

For further details see:

C. Köstler. *A noncommutative extended de Finetti theorem*. J. Funct. Anal. **258**, 1073-1120 (2010)

## Intermediate discussion

- Noncommutative conditional independence emerges from distributional symmetries in terms of commuting squares
- Exchangeability is too weak to identify the structure of the underlying noncommutative probability space
- All reverse implications in the noncommutative extended de Finetti theorem fail due to deep structural reasons!

For further details see:

C. Köstler. *A noncommutative extended de Finetti theorem*. *J. Funct. Anal.* **258**, 1073-1120 (2010)

## Intermediate discussion

- Noncommutative conditional independence emerges from distributional symmetries in terms of commuting squares
- Exchangeability is too weak to identify the structure of the underlying noncommutative probability space
- All reverse implications in the noncommutative extended de Finetti theorem fail due to deep structural reasons!
- This will become clear from braidability...

For further details see:

C. Köstler. *A noncommutative extended de Finetti theorem*. J. Funct. Anal. **258**, 1073-1120 (2010)

R. Gohm & C. Köstler. *Noncommutative independence from the braid group  $\mathbb{B}_\infty$* . Commun. Math. Phys. **289**, 435-482 (2009)

# Artin braid groups $\mathbb{B}_n$

## Algebraic Definition (Artin 1925)

$\mathbb{B}_n$  is presented by  $n - 1$  generators  $\sigma_1, \dots, \sigma_{n-1}$  satisfying

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i - j| = 1 \quad (\text{B1})$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| > 1 \quad (\text{B2})$$



Figure: Artin generators  $\sigma_i$  (left) and  $\sigma_i^{-1}$  (right)

$\mathbb{B}_1 \subset \mathbb{B}_2 \subset \mathbb{B}_3 \subset \dots \subset \mathbb{B}_\infty$  (inductive limit)

# Braidability

## Definition (Gohm & K. '08)

A sequence  $(x_n)_{n \geq 0}$  in  $(\mathcal{A}, \varphi)$  is **braidable** if there exists a representation  $\rho: \mathbb{B}_\infty \rightarrow \text{Aut}(\mathcal{A}, \varphi)$  satisfying:

$$\begin{aligned}x_n &= \rho(\sigma_n \sigma_{n-1} \cdots \sigma_1) x_0 && \text{for all } n \geq 1; \\x_0 &= \rho(\sigma_n) x_0 && \text{if } n \geq 2.\end{aligned}$$

# Braidability

## Definition (Gohm & K. '08)

A sequence  $(x_n)_{n \geq 0}$  in  $(\mathcal{A}, \varphi)$  is **braidable** if there exists a representation  $\rho: \mathbb{B}_\infty \rightarrow \text{Aut}(\mathcal{A}, \varphi)$  satisfying:

$$\begin{aligned}x_n &= \rho(\sigma_n \sigma_{n-1} \cdots \sigma_1) x_0 && \text{for all } n \geq 1; \\x_0 &= \rho(\sigma_n) x_0 && \text{if } n \geq 2.\end{aligned}$$

## Braidability extends exchangeability

- If  $\rho(\sigma_n^2) = \text{id}$  for all  $n$ , one has a representation of  $\mathbb{S}_\infty$ .

# Braidability

## Definition (Gohm & K. '08)

A sequence  $(x_n)_{n \geq 0}$  in  $(\mathcal{A}, \varphi)$  is **braidable** if there exists a representation  $\rho: \mathbb{B}_\infty \rightarrow \text{Aut}(\mathcal{A}, \varphi)$  satisfying:

$$\begin{aligned}x_n &= \rho(\sigma_n \sigma_{n-1} \cdots \sigma_1) x_0 && \text{for all } n \geq 1; \\x_0 &= \rho(\sigma_n) x_0 && \text{if } n \geq 2.\end{aligned}$$

## Braidability extends exchangeability

- If  $\rho(\sigma_n^2) = \text{id}$  for all  $n$ , one has a representation of  $\mathbb{S}_\infty$ .
- $(x_n)_{n \geq 0}$  is exchangeable  $\Leftrightarrow \begin{cases} (x_n)_{n \geq 0} \text{ is braidable and} \\ \rho(\sigma_n^2) = \text{id for all } n. \end{cases}$

# Braidability implies spreadability

Consider the conditions:

- (a)  $(x_n)_{n \geq 0}$  is exchangeable
- (b)  $(x_n)_{n \geq 0}$  is **braidable**
- (c)  $(x_n)_{n \geq 0}$  is spreadable
- (d)  $(x_n)_{n \geq 0}$  is stationary and  $\mathcal{A}^{\text{tail}}$ -independent

# Braidability implies spreadability

Consider the conditions:

- (a)  $(x_n)_{n \geq 0}$  is exchangeable
- (b)  $(x_n)_{n \geq 0}$  is **braidable**
- (c)  $(x_n)_{n \geq 0}$  is spreadable
- (d)  $(x_n)_{n \geq 0}$  is stationary and  $\mathcal{A}^{\text{tail}}$ -independent

Theorem (Gohm & K. '08)

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d), \quad \text{but } (a) \not\Leftarrow (b) \not\Leftarrow (d)$$

# Braidability implies spreadability

Consider the conditions:

- (a)  $(x_n)_{n \geq 0}$  is exchangeable
- (b)  $(x_n)_{n \geq 0}$  is **braidable**
- (c)  $(x_n)_{n \geq 0}$  is spreadable
- (d)  $(x_n)_{n \geq 0}$  is stationary and  $\mathcal{A}^{\text{tail}}$ -independent

Theorem (Gohm & K. '08)

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d), \quad \text{but } (a) \not\Leftarrow (b) \not\Leftarrow (d)$$

## Observation

Large and interesting class of spreadable sequences is obtained from braidability.

# Intermediate Discussion

There are many examples of braidability!

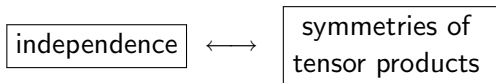
- subfactor inclusion with small Jones index ('Jones-Temperley-Lieb algebras and Hecke algebras')
- left regular representation of  $\mathbb{B}_\infty$
- vertex models in quantum statistical physics ('Yang-Baxter equations')
- ...
- representations of the symmetric group  $\mathbb{S}_\infty$

For further details see:

R. Gohm & C. Köstler. *Noncommutative independence from the braid group*  $\mathbb{B}_\infty$ . Commun. Math. Phys. **289**, 435–482 (2009)

# Is there a free analogue of de Finetti's theorem?

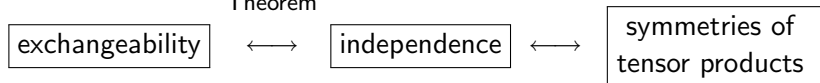
## Classical Probability



# Is there a free analogue of de Finetti's theorem?

## Classical Probability

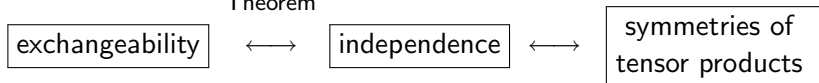
De Finetti's  
Theorem



# Is there a free analogue of de Finetti's theorem?

## Classical Probability

De Finetti's  
Theorem

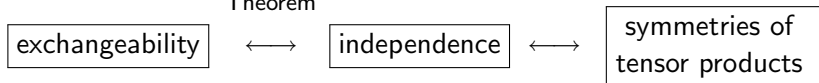


↔ Subject of distributional symmetries and invariance principles

# Is there a free analogue of de Finetti's theorem?

## Classical Probability

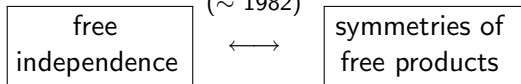
De Finetti's  
Theorem



↔ Subject of distributional symmetries and invariance principles

## Free Probability

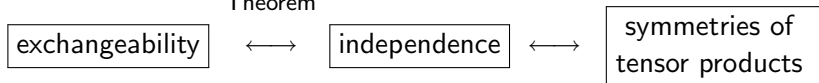
Voiculescu  
(~ 1982)



# Is there a free analogue of de Finetti's theorem?

## Classical Probability

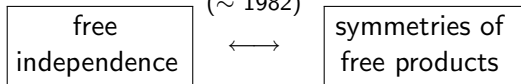
De Finetti's  
Theorem



↪ Subject of distributional symmetries and invariance principles

## Free Probability

Voiculescu  
(~ 1982)

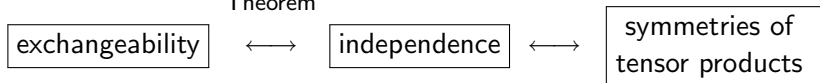


↪ Foundation of free probability theory

# Is there a free analogue of de Finetti's theorem?

## Classical Probability

De Finetti's  
Theorem

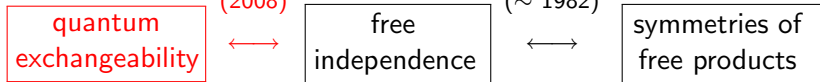


↪ Subject of distributional symmetries and invariance principles

## Free Probability

K. & S.  
(2008)

Voiculescu  
(~ 1982)

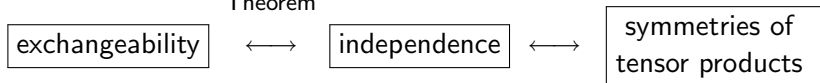


↪ Foundation of free probability theory

# Is there a free analogue of de Finetti's theorem?

## Classical Probability

De Finetti's  
Theorem

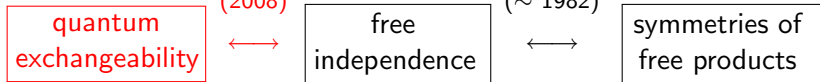


↪ Subject of distributional symmetries and invariance principles

## Free Probability

K. & S.  
(2008)

Voiculescu  
(~ 1982)



↪ Foundation of free probability theory

↪ New direction of research in free probability

## Key idea (K. & Speicher '08)

- Rewrite exchangeability (for a classical sequence  $X$ ) as a co-symmetry using the Hopf  $C^*$ -algebra  $C(\mathbb{S}_k)$

## Key idea (K. & Speicher '08)

- Rewrite exchangeability (for a classical sequence  $X$ ) as a co-symmetry using the Hopf  $C^*$ -algebra  $C(\mathbb{S}_k)$
- Apply 'quantization':

## Key idea (K. & Speicher '08)

- Rewrite exchangeability (for a classical sequence  $X$ ) as a co-symmetry using the Hopf  $C^*$ -algebra  $C(\mathbb{S}_k)$
- Apply 'quantization':
  - ◇ Replace  $X_k$ 's by **operators**  $x_1, x_2, \dots$

## Key idea (K. & Speicher '08)

- Rewrite exchangeability (for a classical sequence  $X$ ) as a co-symmetry using the Hopf  $C^*$ -algebra  $C(\mathbb{S}_k)$
- Apply 'quantization':
  - ◇ Replace  $X_k$ 's by **operators**  $x_1, x_2, \dots$
  - ◇ Replace  $C(\mathbb{S}_k)$  by  
**Wang's quantum permutation group  $A_s(k)$**

## Key idea (K. & Speicher '08)

- Rewrite exchangeability (for a classical sequence  $X$ ) as a co-symmetry using the Hopf  $C^*$ -algebra  $C(\mathbb{S}_k)$
- Apply 'quantization':
  - ◇ Replace  $X_k$ 's by **operators**  $x_1, x_2, \dots$
  - ◇ Replace  $C(\mathbb{S}_k)$  by  
**Wang's quantum permutation group  $A_s(k)$**

For details see:

C. K\"ostler & R. Speicher. *A noncommutative de Finetti theorem: Invariance under quantum permutations is equivalent to freeness with amalgamation*. Commun. Math. Phys. **291**(2), 473–490 (2009)

## Key idea (K. & Speicher '08)

- Rewrite exchangeability (for a classical sequence  $X$ ) as a co-symmetry using the Hopf  $C^*$ -algebra  $C(\mathbb{S}_k)$
- Apply 'quantization':
  - ◇ Replace  $X_k$ 's by **operators**  $x_1, x_2, \dots$
  - ◇ Replace  $C(\mathbb{S}_k)$  by  
**Wang's quantum permutation group  $A_s(k)$**

For details see:

C. K\"ostler & R. Speicher. *A noncommutative de Finetti theorem: Invariance under quantum permutations is equivalent to freeness with amalgamation*. Commun. Math. Phys. **291**(2), 473–490 (2009)

S. Curran. *Quantum exchangeable sequences of algebras* Indiana Univ. Math. J. 58 (2009), pp. 1097–1125.

S. Curran *Quantum rotatability*. Trans. Amer. Math. Soc. 362 (2010), 4831–4851.

# A free analogue of de Finetti's theorem

## Theorem (K. & Speicher 2008)

The following are equivalent for an infinite sequence of random variables  $x_1, x_2, \dots$  in  $(\mathcal{A}, \varphi)$ :

# A free analogue of de Finetti's theorem

## Theorem (K. & Speicher 2008)

The following are equivalent for an infinite sequence of random variables  $x_1, x_2, \dots$  in  $(\mathcal{A}, \varphi)$ :

- (a) the sequence is **quantum exchangeable**

# A free analogue of de Finetti's theorem

## Theorem (K. & Speicher 2008)

The following are equivalent for an infinite sequence of random variables  $x_1, x_2, \dots$  in  $(\mathcal{A}, \varphi)$ :

- (a) the sequence is **quantum exchangeable**
- (b) the sequence is identically distributed and **freely independent with amalgamation over  $\mathcal{T}$**

Here  $\mathcal{T}$  denotes the **tail von Neumann algebra**

$$\mathcal{T} = \bigcap_{n \in \mathbb{N}} \text{vN}(x_k | k \geq n)$$

# A free analogue of de Finetti's theorem

## Theorem (K. & Speicher 2008)

The following are equivalent for an infinite sequence of random variables  $x_1, x_2, \dots$  in  $(\mathcal{A}, \varphi)$ :

- (a) the sequence is **quantum exchangeable**
- (b) the sequence is identically distributed and **freely independent with amalgamation over  $\mathcal{T}$**   
(in the sense of Voiculescu 1985)

Here  $\mathcal{T}$  denotes the **tail von Neumann algebra**

$$\mathcal{T} = \bigcap_{n \in \mathbb{N}} \text{vN}(x_k | k \geq n)$$

# A free analogue of de Finetti's theorem

## Theorem (K. & Speicher 2008)

The following are equivalent for an infinite sequence of random variables  $x_1, x_2, \dots$  in  $(\mathcal{A}, \varphi)$ :

- (a) the sequence is **quantum exchangeable**
- (b) the sequence is identically distributed and **freely independent with amalgamation over  $\mathcal{T}$**   
(in the sense of Voiculescu 1985)
- (c) the sequence canonically embeds into  $\star_{\mathcal{T}}^{\mathbb{N}} \text{vN}(x_1, \mathcal{T})$ , a von Neumann algebraic **amalgamated free product over  $\mathcal{T}$**

Here  $\mathcal{T}$  denotes the **tail von Neumann algebra**

$$\mathcal{T} = \bigcap_{n \in \mathbb{N}} \text{vN}(x_k | k \geq n)$$

# Summary

- Out of the noncommutative extended de Finetti theorem emerges a very general notion of noncommutative conditional independence which invites to develop several new lines of research along the classical subject of distributional symmetries and invariance principles.

# Summary

- Out of the noncommutative extended de Finetti theorem emerges a very general notion of noncommutative conditional independence which invites to develop several new lines of research along the classical subject of distributional symmetries and invariance principles.
- Braidability as a new symmetry extends exchangeability and provides a new perspective on subfactor theory and free probability theory

# Summary

- Out of the noncommutative extended de Finetti theorem emerges a very general notion of noncommutative conditional independence which invites to develop several new lines of research along the classical subject of distributional symmetries and invariance principles.
- Braidability as a new symmetry extends exchangeability and provides a new perspective on subfactor theory and free probability theory
- Quantum exchangeability gives a beautiful characterization of freeness with amalgamation and opens a new research direction in free probability. The free version of the de Finetti theorem shows that a quantum symmetry leads to a quantum probabilistic invariant.

# Summary

- Out of the noncommutative extended de Finetti theorem emerges a very general notion of noncommutative conditional independence which invites to develop several new lines of research along the classical subject of distributional symmetries and invariance principles.
- Braidability as a new symmetry extends exchangeability and provides a new perspective on subfactor theory and free probability theory
- Quantum exchangeability gives a beautiful characterization of freeness with amalgamation and opens a new research direction in free probability. The free version of the de Finetti theorem shows that a quantum symmetry leads to a quantum probabilistic invariant.
- A new approach is opened to the theory of asymptotic representations of symmetric groups.
- ...

Thank you!

# References

C. Köstler. *A noncommutative extended de Finetti theorem*. J. Funct. Anal. **258**, 1073–1120 (2010)

(electronic: [arXiv:0806.3621v1](https://arxiv.org/abs/0806.3621v1))

R. Gohm & C. Köstler. *Noncommutative independence from the braid group  $\mathbb{B}_\infty$* . Commun. Math. Phys. **289**, 435–482 (2009)

(electronic: [arXiv:0806.3691v2](https://arxiv.org/abs/0806.3691v2))

C. Köstler & R. Speicher. *A noncommutative de Finetti theorem: Invariance under quantum permutations is equivalent to freeness with amalgamation*.

Commun. Math. Phys. **291**(2), 473–490 (2009)

(electronic: [arXiv:0807.0677v1](https://arxiv.org/abs/0807.0677v1))

C. Köstler. *On Lehner's 'free' noncommutative analogue of de Finetti's theorem*. 11 pages. To appear in Proc. Amer. Math. Soc.

(electronic: [arXiv:0806.3632v1](https://arxiv.org/abs/0806.3632v1))

R. Gohm & C. Köstler. *Noncommutative independence from characters of the symmetric group  $\mathbb{S}_\infty$* . 47 pages. Preprint (2010).

(electronic: [arXiv:1005.5726](https://arxiv.org/abs/1005.5726))