Spectral Triples, Aperiodic Tilings and Maximal Equicontinuous Factors

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Introduction

What can spectral triples tell us about aperiodic order?

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There has been a lot of activity in constructing spectral triples for spaces (commutative spectral triples).

- manifolds, fractals, embedded Cantor sets, ultra metric Cantor sets (Pearson, Bellissard).
- The discrete hull of an aperiodic tiling is a ultra metric Cantor set.
- Aperiodic order is modeled by aperiodic tilings (quasicrystals).

Motivation: aperiodic media (solid state physics).
Modeled by aperiodic Delone sets (set of atomic positions) $D \subset \mathbb{R}^n$ and best visualised by tilings T.
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- Diffraction: D is ordered if its diffraction spectrum is pure point (Fourier transform of the autocorrelation of $\sum_{x\in D} \delta_x$ is a pure point measure).

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- Topology: T has low topological complexity if $\dim K(A_T) \otimes \mathbb{Q}$ is finite.

Aperiodic order

Combinatorial complexity, patch counting function. Let $R > 0$, $patch_R(T, x) = \{$ tiles of T meeting $B_R(x)$ $\}$ (R-patch of T at x).

$$p(R) = \#\{R\text{-patches up to translation occuring in } T\}$$

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Minimal complexity: $\beta = n$.

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repetitivity exponent γ: $F(R) \sim cR^\gamma$.
$F(R)$ sub-linear $\Rightarrow$ T is periodic [Lagarias].
$\gamma = 1$ linear repetitive .

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Canonical cut & project [Antoine Julien, 2009]:
minimal combinatorial complexity $\Leftrightarrow$ low topological complexity.

Algebras from tilings

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- There is a further non-canonical reduction to $X_T \subset \Xi$ yielding a $\mathbb{Z}^n$ action α and $C(X_T) \rtimes_\alpha \mathbb{Z}^n$.

The three non-commutative algebras are strongly Morita equivalent.

Tiling dynamcial systems

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Ξ is metrisable. Tiling people like to choose the following tiling metric

$$d(T, T') = \inf\{\frac{1}{R+1} : patch_R(T, 0) = patch_R(T', 0)\}$$

but there is nothing canonical about it.

Let $\mathcal{A}$ be a C^*-algebra. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ for $\mathcal{A}$ is

1. a (faithful) reps π of $\mathcal{A}$ as an algebra of bounded operators on a Hilbert space $\mathcal{H}$
2. an unbounded selfadjoint operator D on $\mathcal{H}$ (the Dirac) such that
   - $\mathcal{B} := \{a \in \mathcal{A} : \|[D, \pi(a)]\| < \infty\}$ is dense in $\mathcal{A}$,
   - $(D + i)^{-1}$ is compact.
   - further conditions (according to taste) (differential operator of degree 1, Poincaré duality, finite dimensional, ..., for tilings?)
   - (even case) a $\mathbb{Z}_2$-grading on $\mathcal{H}$: an operator Γ on $\mathcal{H}$, $\Gamma^* = \Gamma^{-1} = \Gamma$ such that $[D, \Gamma]_+ = 0$, $[\Gamma, \pi(a)] = 0$.

1. Spectral distance:

$$\mathrm{d}_s(\chi_1, \chi_2) := \sup_{a \in \mathcal{A}} \{ |\chi_1(a) - \chi_2(a)| : \|[D, \pi(a)]\| \leq 1 \}.$$

defines a distance on the state space of $\mathcal{A}$.

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2 Zeta function: Given a trace Tr and a compact operator K we can ask, when, and on which domain of $\mathbb{C}$,

$$s \mapsto \zeta_K(s) := \mathrm{Tr}(|K|^s)$$

is well defined / analytic / meromorphic ... Taking $K = D^{-1}$ we obtain the zeta function of the spectral triple

$$\zeta(s) = \zeta_{D^{-1}}(s)$$

The left most value $s_0 \in \mathbb{R}$ such that ζ is analytic on $\Re(s) > s_0$ spectral dimension of the spectral triple.

3 ...

Rieffel: (X,d) compact metric space. $\mathcal{H} = L^2(X \times X \backslash diag, m)$, m symmetric measure.

$$\pi(f)\psi(x,y) = f(x)\psi(x,y)$$

$$D\psi(x,y) = \frac{1}{d(x,y)}\psi(y,x)$$

Then

$$\|[D,\pi(f)]\| = ess - \sup_{(x,y)} \frac{|f(x)-f(y)|}{d(x,y)} = \|f\|_{ess-Lip}$$

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But when is D compact?

Note that this is actually an even spectral triple in disguise.

Christensen, Ivan: ”sum’s of two-dimensional spectral triples” (variant)

Construct a graph $\mathcal{G} = \bigcup_n \mathcal{G}_n$. Vertices $\mathcal{G}_n^{(0)} \subset \mathcal{G}_{n+1}^{(0)} \subset X$ form a chain of Delone sets: There are strictly decreasing null-sequences of positive numbers $(r_n)_n$, $(R_n)_n$:

- r_n is the minimal distance between points in $\mathcal{G}_n^{(0)}$
- R_n is the diameter of the largest ball in $X \backslash \mathcal{G}_n^{(0)}$

In particular $\mathcal{G}^{(0)}$ is dense in X.

Edges are defined between neighbours: $(x, y) \in \mathcal{G}_n^{(1)}$ if $d(x, y) \leq \ell R_n$ where $\ell \geq 1$ is some fixed number.

$$m = \sum_{(x,y)\in\mathcal{G}^{(1)}} (\delta_{x,y} + \delta_{y,x})$$

Lemma

If $\ell \geq 2$ then d_s is Lipschitz equivalent to d.

Spectral Triple for the discrete hull of a tiling

Pearson, Bellissard: spectral triple for an ultra metric Cantor set (X, d) (variant).

An ultra-metric has the property that for $r > 0$ either $B_r(x) = B_r(y)$ or $B_r(x) \cap B_r(y) = \emptyset$. In particular there is a unique r-covering of X.

X being Cantor implies that im$d$ corresponds to the values of a strictly decreasing null-sequence $(\delta_n)_n$. This gives rise to a weighted Michon tree (a Bratteli diagram of $C(X)$), the weights being the diameters of the cylinder sets of R-patches.

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This can be applied to Ξ or to X

Lemma (Julien, Savinien)

Let T be a primitive substitution tiling. The spectral dimension of the above spectral triples is the complexity exponent of the tiling.

Why that metric?

Could replace the weights by something more inherent: frequencies.

Lemma (Durand)

If T is linear repetitive frequencies as weights yield a spectral distance which is Lipschitz equivalent to d.

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- $\mathcal{H}_\alpha = \ell^2(\mathbb{Z}^n, \mathcal{H})$
- $\pi_\alpha(f)\psi(n) = \pi(f \circ \alpha^n)$
- $D_\alpha = \sigma_x D + \sigma_z \hat{q}$ where $\hat{q}\psi(n) = n\psi(n)$ and $\sigma_x\sigma_z + \sigma_z\sigma_x = 0$.

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Whereas D_α has still complact resolvent,
$\mathcal{B} := \{a \in C(X_T) \rtimes_\alpha \mathbb{Z}^n : \|[D_\alpha, \pi_\alpha(a)]\| < \infty$ is no longer dense,
except if d is α-invariant.
This is never the case for aperiodic repetitive tilings!

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What is $\mathcal{B}$?

Maximal equicontinuous factor

Generalities

Let (X, α) be a compact metrisable $\mathbb{Z}^n$-dynamical system. It is equicontinuous if $\{\alpha^k : k \in \mathbb{Z}^n\}$ is an equicontinuous family (there exists a compatible α-invariant metric on X).
There is always a maximal equicontinuous factor

$$(X, \alpha) \xrightarrow{\mu} (G, \rho)$$

(μ is a $\mathbb{Z}^n$ -equivariant continuous surjection, the factor a universal reppellent object).
Auslander: The maximal continuous factor (G, ρ) is a compact abelian group G with action ρ by rotation.

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For tilings the maximal equicontinuous factor is a well known object. For a Sturmian sequence with slope θ, $G = S^1$ and ρ is rotation by θ.

If μ is almost one-to-one, the dynamical spectrum of (X, α) and (G, ρ) coincide. This is the case for canonical cut & project tilings (for these $G = \mathbb{T}^m$ (m is the codimension).

Recall $\mathcal{B} = \{a \in C(X) \rtimes_\alpha \mathbb{Z}^n : \|[D_\alpha, \pi_\alpha(a)]\| < \infty\}$.

Theorem

Consider the above situation

$$(X, \alpha, d) \xrightarrow{\mu} (G, \rho, \delta)$$

where d and δ are compatible metrics on X and G, the latter invariant. If μ is Lipschitz continuous then $\mathcal{B} = \mu^*(C(G)_{Lip} \rtimes_\rho \mathbb{Z}^n)$.

Maximal equicontinuous factor
Non commutative description

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When is μ Lipschitz continuous?

Consider a tiling dynamical system (X_T, α)

- choose d the "standard" tiling metric on X_T
- suppose $G = \mathbb{T}^m$
- choose δ euclidean metric on G

Theorem

For canonical cut & project tilings μ is Lipschitz continuous if T is linearly repetitive.

For Sturmian sequences the converse holds as well.

Either stick to the choice of metrics then this *conjecturally* characterises linear repetitivity, or choose a better metric d.

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- The spectral triples for tilings considered carry information about the combinatorial complexity.
- The induced triples describe the maximal equicontinuous factor.
- They are sensitive to the repetitivity properties of the tiling.
- The latter are sensitive to the number theoretical properties of the cut & project data.