Quantum Homogeneous Spaces

Paweł Kasprzak

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Introduction

- Quantum groups $\rightsquigarrow$ quantum homogeneous spaces
- Podleś spheres $\rightsquigarrow$ generically non-quotient
- Compact case $\rightsquigarrow$ C^*-algebraic ergodicity
- Non-compact case $\rightsquigarrow$ C^*-algebraic approach - not enough

Definition

$\mathbb{G} = (M, \Delta)$ - locally compact quantum group (LCQG):

- M - von Neumann algebra;
- $\Delta : M \to M \otimes M$ - normal, unital, coassociative
  - $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$
- $\exists\ \phi$, ψ - left and right Haar weights on M
  - $(\mathrm{id} \otimes \phi)\Delta(x) = \phi(x)1$
  - $(\psi \otimes \mathrm{id})\Delta(x) = \psi(x)1$

  for all $x \in \mathrm{M}_+$.

Notation

- $M = \mathrm{L}^\infty(\mathbb{G})$, $\Delta = \Delta_{\mathbb{G}}$
- $\mathbb{G} \rightsquigarrow \mathrm{C}_0(\mathbb{G})$, $\Delta_{\mathbb{G}} \in \mathrm{Mor}(\mathrm{C}_0(\mathbb{G}), \mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{G}))$
- C^***-algebraic version - visible on the von Neumann level!**

Actions of $\mathbb{G}$

Definition

- N - von Neumann algebra
- $\alpha : N \to \mathrm{L}^\infty(\mathbb{G}) \otimes N$ - left action of $\mathbb{G}$ on N:
  - $(\mathrm{id} \otimes \alpha)\alpha = (\Delta_{\mathbb{G}} \otimes \mathrm{id})\alpha$
- Define: $N^\alpha = \{x \in N \,|\, \alpha(x) = 1 \otimes x\}$
- α - ergodic if $N^\alpha = \mathbb{C}1$

Example

Example

- G - locally compact group
- X - G - homogeneous space
- $N = \mathrm{L}^\infty(\mathrm{X})$
- $\Delta_\mathrm{X} : \mathrm{L}^\infty(\mathrm{X}) \to \mathrm{L}^\infty(\mathrm{G}) \otimes \mathrm{L}^\infty(\mathrm{X})$
  - $\Delta_\mathrm{X}(f)(g, x) = f(gx)$
- Δ_N - ergodic
- Can one detect $(\mathrm{C}_0(\mathrm{X}), \Delta_\mathrm{X})$ inside $(\mathrm{L}^\infty(\mathrm{X}), \Delta_\mathrm{X})$?

Continuous actions of $\mathbb{G}$

Definition

$\mathbb{D} = (D, \Delta_D)$ is a $\mathbb{G}$-C*-algebra:

- D - C*-algebra
- $\Delta_D \in \mathrm{Mor}(D, \mathrm{C}_0(\mathbb{G}) \otimes D)$ is a continuous action of $\mathbb{G}$:
  - $(\mathrm{id} \otimes \Delta_D)\Delta_D = (\Delta_{\mathbb{G}} \otimes \mathrm{id})\Delta_D$
  - $\{\Delta_D(d)(a \otimes 1) |\ d \in \mathrm{D},\ a \in \mathrm{C}_0(\mathbb{G})\}^{cls} = \mathrm{C}_0(\mathbb{G}) \otimes D$
- Notation: $D = \mathrm{C}_0(\mathbb{D})$, $\Delta_D = \Delta_{\mathbb{D}}$

Remark

1. G - locally compact group, $\mathbb{G} = (\mathrm{C}_0(\mathrm{G}), \Delta)$
2. $\mathbb{G}$ - C*-algebras $\leftrightsquigarrow$ G - C*-algebras
3. $\gamma_g(d) = (\mathrm{ev}_{g^{-1}} \otimes \mathrm{id})\Delta_{\mathbb{D}}(d)$
4. $\mathrm{ev}_g(f) = f(g)$ for any $f \in \mathrm{C}_0(\mathrm{G})$

Characterization of $(\mathrm{C}_0(\mathrm{X}), \Delta_\mathrm{X})$

1. Let $x \in \mathrm{X}$ and $\mathrm{G}_x \subset \mathrm{G}$ - stabilizer of x
2. $\mathrm{X} \cong \mathrm{G}\,/\,\mathrm{G}_x$
3. $\mathrm{L}^\infty(\mathrm{G}\,/\,\mathrm{G}_x) \cong \{x \in \mathrm{L}^\infty(\mathrm{G}) : xR_g = R_g x \text{ for all } g \in \mathrm{G}_x\}$

Theorem (Vaes)

- $\mathbb{G}$ - *LCQG + regular*
- $\mathbb{G}_0$ - *closed quantum subgroup of* $\mathbb{G}$
- $(\mathrm{L}^\infty(\mathbb{G}/\mathbb{G}_0), \Delta_{\mathbb{G}/\mathbb{G}_0})$ - *not difficult to construct*
- $\exists!$ $\mathbb{G}$ - C^**-algebra* $\mathbb{G}/\mathbb{G}_0$ - *difficult to construct:*
  - $\mathrm{C}_0(\mathbb{G}/\mathbb{G}_0) \subset \mathrm{L}^\infty(\mathbb{G}/\mathbb{G}_0)$ - *strongly dense*
  - $\mathbb{G}$-C^**-algebra coincides with restriction of* $\Delta_{\mathbb{G}/\mathbb{G}_0}$
  - $\Delta_{\mathbb{G}/\mathbb{G}_0}(\mathrm{L}^\infty(\mathbb{G}/\mathbb{G}_0)) \subset \mathrm{M}(\mathcal{K}(\mathrm{L}^2(\mathbb{G})) \otimes \mathrm{C}_0(\mathbb{G}/\mathbb{G}_0))$
  - $\Delta_{\mathbb{G}/\mathbb{G}_0} : \mathrm{L}^\infty(\mathbb{G}/\mathbb{G}_0)) \to \mathrm{M}(\mathcal{K}(\mathrm{L}^2(\mathbb{G})) \otimes \mathrm{C}_0(\mathbb{G}/\mathbb{G}_0))$ *is strict.*

Strictness of $\Delta_{\mathbb{G}/\mathbb{G}_0}$

Strictness of $\Delta_{\mathbb{G}/\mathbb{G}_0}$:

- $x_i \in \mathrm{L}^\infty(\mathbb{G}/\mathbb{G}_0)$ - uniformly bounded $*$ - strongly convergent
- $y \in \mathcal{K}(\mathrm{L}^2(\mathbb{G})) \otimes \mathrm{C}_0(\mathbb{G}/\mathbb{G}_0)$
- The net $\Delta_{\mathbb{G}/\mathbb{G}_0}(x_i)y$ - norm convergent

Quantum homogeneous space <u>are not</u> necessarily of the quotient type!

1. Podleś spheres
2. Rieffel deformation of homogeneous spaces

Quantum homogeneous spaces

Definition

- N - von Neumann algebra
- $\Delta_N : N \to \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} N$ - ergodic action of $\mathbb{G}$
- We say that (N, Δ_N) is a quantum homogeneous space if:
  - $\exists$ $\mathbb{G}$-C^*-algebra $\mathbb{D}$, $\mathrm{C}_0(\mathbb{D}) \subset N$ - strongly dense
  - Δ_N restricts to $\Delta_{\mathbb{D}}$
  - $\Delta_N(N) \subset \mathrm{M}(\mathcal{K}(\mathrm{L}^2(\mathbb{G})) \otimes \mathrm{C}_0(\mathbb{D}))$
  - $\Delta_N : N \to \mathrm{M}(\mathcal{K}(\mathrm{L}^2(\mathbb{G})) \otimes \mathrm{C}_0(\mathbb{D}))$ is strict

Proposition

- $\mathbb{D}$ *is uniquely determined by* (N, Δ_N)
- (N, Δ_N) *is uniquely determined by* $\mathbb{D}$
- *Notation:* $(N, \Delta_N) = (\mathrm{L}^\infty(\mathbb{D}), \Delta_{\mathrm{L}^\infty(\mathbb{D})})$

Classical case

Theorem

- G - *locally compact group,* $\mathbb{G} = (L^{\infty}(G), \Delta)$
- N - *commutative von Neumann algebra*
- (N, Δ_N) - *quantum homogeneous space*
- $\mathbb{D}$ - C^**-algebraic version*
- *Then* $\mathrm{Sp}(\mathbb{D})$ - *is a* G*-homogeneous space*
- $N = L^{\infty}(\mathrm{Sp}(\mathbb{D}))$

This establishes a 1-1 correspondence between QHS with N - commutative and G - homogeneous spaces.

Classical case - proof

Proof.

- $\mathbb{D}$ is a $\mathbb{G}$-C^*-algebra $\leftrightsquigarrow$ $\mathrm{Sp}(\mathbb{D})$ is a G-space
- $y \in \mathrm{Sp}(\mathbb{D})$, define $\pi_{\mathbb{D}} \in \mathrm{Mor}(C_0(\mathbb{D}), C_0(G))$:
- $\pi_{\mathbb{D}}(d) = (\mathrm{id} \otimes \mathrm{ev}_y)\Delta_{\mathbb{D}}(d)$
- $\Delta_N(N) \subset M(\mathcal{K}(L^2(\mathbb{G})) \otimes C_0(\mathbb{D})) \rightsquigarrow \pi_N(x) = (\mathrm{id} \otimes \mathrm{ev}_y)\Delta_N(x)$
- $\pi_N : N \to L^\infty(G)$ is a normal injective $*$-homomorphism
- $\pi_{\mathbb{D}} \rightsquigarrow$ G-map $\pi_{\mathrm{Sp}(\mathbb{D})} : G \to \mathrm{Sp}(\mathbb{D})$
- $\pi_{\mathbb{D}}$ - injective $\rightsquigarrow$ $\pi_{\mathrm{Sp}(\mathbb{D})}$ - dense image
- $\pi_{\mathbb{D}}$ - surjective $\Rightarrow$ $\mathrm{Sp}(\mathbb{D})$ is homogeneous

□

Classical case - proof

Proof.

- $y' \notin \pi_{\mathrm{Sp}(\mathbb{D})}(\mathrm{G})$
- $\mathcal{U}$ - compact ngbhd of $e \in \mathrm{G}$ of $vol(\mathcal{U}) = 1$
- $\mathcal{U}y' \subset \mathrm{Sp}(\mathbb{D})$ - compact subset
- $\tilde{\mathcal{O}}_i \subset \mathrm{Sp}(\mathbb{D})$ - net of open sets converging to $\mathcal{U}y'$
- $\mathcal{O}_i = \pi^{-1}_{\mathrm{Sp}(\mathbb{D})}(\tilde{\mathcal{O}}_i) \subset \mathrm{G}$ - net of open subsets
- $\chi_{\mathcal{O}_i} \in N$ - unif. bnd. $*$-strongly convergent to 0
- $\Delta_N(\chi_{\mathcal{O}_i})(|\chi_{\mathcal{U}} >< \chi_{\mathcal{U}}| \otimes f)$ - norm convergent to 0
- $\lim_i(\mathrm{id} \otimes \mathrm{ev}_{y'})\Big(\Delta_N(\chi_{\mathcal{O}_i})(|\chi_{\mathcal{U}} >< \chi_{\mathcal{U}}| \otimes f)\Big) =$
  $= |\chi_{\mathcal{U}} >< \chi_{\mathcal{U}}| f(y') \neq 0$ - contradiction

□

Compact case

Definition

- $\mathbb{G} = (A, \Delta)$
- A - unital C*-algebra
- $\Delta \in \mathrm{Mor}(A, A \otimes A)$ - comultiplication
- $[\Delta(A)(A \otimes 1)] = A \otimes A = [\Delta(A)(1 \otimes A)]$

Theorem

There exists a unique left and right invariant Haar state $\phi : A \to \mathbb{C}$

We shall assume that ϕ is faithful.

Compact case

Definition

- $\mathbb{D}$ - unital $\mathbb{G}$-C^*-algebra
- $\mathbb{D}$ is a QHSP if $\Delta_{\mathbb{D}}$ is ergodic

1. It is purely C^*-algebraic concept
2. Assumption: $\Delta_{\mathbb{D}}$ - injective

Theorem

- *There exists a QHS* $(\mathrm{L}^\infty(\mathbb{D}), \Delta_{\mathrm{L}^\infty(\mathbb{D})})$ *whose* C^**-algebraic version coincides with* $\mathbb{D}$
- N *- von Neumann algebra,* $\Delta_N : N \to \mathrm{L}^\infty(\mathbb{G}) \otimes N$ *- ergodic. There exists* $\mathbb{D}$ *such that* (N, Δ_N) *is a QHS.*

Rieffel deformation of homogeneous spaces

- X - homogeneous G-space,
- $\mathbb{G} = (C_0(G), \Delta_G)$, $\mathbb{X} = (C_0(X), \Delta_X)$
- $\Gamma \subset G$ - closed abelian subgroup
- Ψ - 2-cocycle on $\hat{\Gamma}$
- $\mathbb{G}^\Psi$ - Rieffel deformation of $\mathbb{G}$
- $\mathbb{X} \rightsquigarrow \mathbb{X}^\Psi$?

Deformation procedure of X

1. construct the crossed product $\Gamma \ltimes \mathsf{L}^\infty(\mathsf{X})$
2. $\hat{\rho}$ - dual action
3. $\mathsf{L}^\infty(\mathsf{X})$ - $\hat{\rho}$-invariants
4. twist the dual action $\hat{\rho} \rightsquigarrow \hat{\rho}^\Psi$
5. define: $\mathsf{L}^\infty(\mathbb{X}^\Psi)$ - invariants of $\hat{\rho}^\Psi$.

Theorem

There exists an ergodic action

$$\Delta_{\mathsf{L}^\infty(\mathbb{X}^\Psi)} : \mathsf{L}^\infty(\mathbb{X}^\Psi) \to \mathsf{L}^\infty(\mathbb{G}^\Psi) \otimes \mathsf{L}^\infty(\mathbb{X}^\Psi)$$

s.t. $\left(\mathsf{L}^\infty(\mathbb{X}^\Psi), \Delta_{\mathsf{L}^\infty(\mathbb{X}^\Psi)}\right)$ *is a QHS.*

Arxiv

- *Quantum Homogeneous Spaces* arXiv:1007.2438
- **THANK YOU**