Spectral Triples and an Index Pairing for Super-Conformal Nets

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based on joint work with Sebastiano Carpi

Conformal Nets and Spectral Triples

Model Analysis: Super-Current Algebra Models

The General Index Pairing

Conformal Nets and Spectral Triples

Definition: a *local conformal net over* $\mathbb{S}$

is a family $\mathcal{A} = \{\mathcal{A}(I) \subset B(\mathcal{H}) : I \in \mathcal{I}\}$ of von Neumann algebras satisfying

(i) $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$, if $I_1 \subset I_2$, $I_1, I_2 \in \mathcal{I}$

(ii) $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)'$, if $I_1 \cap I_2 = \emptyset$, $I_1, I_2 \in \mathcal{I}$

(iii) covariance wrt. $\mathrm{PSL}(2, \mathbb{R})$

(iv) the conformal Hamiltonian L_0 generating the rotation subgroup has positive spectrum

(v) there exists a unique vacuum vector

Definition: a *graded-local conformal net over* $\mathbb{S}$

is a family $\mathcal{A} = \{\mathcal{A}(I) \subset B(\mathcal{H}) : I \in \mathcal{I}\}$ of von Neumann algebras with common $\mathbb{Z}/2$-grading $\mathrm{Ad}(\Gamma)$ satisfying

(i) $\mathcal{A}(I_1) \subset \mathcal{A}(I_2), \text{ if } I_1 \subset I_2,\ I_1, I_2 \in \mathcal{I}$

(ii) $\mathcal{A}(I_1) \subset Z\mathcal{A}(I_2)'Z, \text{ if } I_1 \cap I_2 = \emptyset,\ I_1, I_2 \in \mathcal{I}$, where $Z = \frac{\mathbf{1}+\mathrm{i}\,\Gamma}{1+\mathrm{i}}$

(iii) covariance wrt. $\mathrm{PSL}(2,\mathbb{R})^{(2)}$

(iv) the conformal Hamiltonian L_0 generating the rotation subgroup has positive spectrum

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Representations

- A *locally normal representation* of $\mathcal{A}$ is a family $\pi = \{\pi_I : \mathcal{A}(I) \to B(\mathcal{H}_\pi) : \ I \in \mathcal{I}\}$ of normal representations on some common separable $\mathcal{H}_\pi$ such that

$$\pi_J|_{\mathcal{A}(I)} = \pi_I \ \text{ if } I \subset J,$$

- Fix $I_0 \in \mathcal{I}$. Then there is $\rho \simeq \pi$ *localised in* I_0, i.e. $\rho_I = \mathbf{1}$ for $I \subset I_0'$, and $\rho_I \in \mathrm{End}(\mathcal{A}(I))$ for $I \supset I_0$.
- Conformal/diffeomorphism covariance...

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Example: super-current algebra nets (sketch)

- Consider central extension of loop group, $\hat{\mathrm{L}}\,\mathrm{U}(1)_{N_i}$, at level $N \in 2\mathbb{N}^d$, generating lattice $L_N \subset \mathbb{R}^d$. Define graded net

$$\mathcal{A}(I) := \bigotimes_{i=1}^{d} \pi_0(\mathrm{L}_I\,\mathrm{U}(1)_{N_i})'' \otimes \mathcal{F}_d(I).$$

- 0-connex image of smeared fields "$J(f) := \sum_{a,n} f_n^a J_n^a$" of super-current algebra generated by J_n^a, F_r^a with

$$[J_m^a, J_n^b] = m\delta_{a,b}\delta_{m,-n}, \quad [F_r^a, F_s^b] = \delta_{a,b}\delta_{r,-s}, \quad [J_m^a, F_r^b] = 0,$$

$a, b = 1, ..., d$, $n \in \mathbb{Z}$, $r \in \frac{1}{2}\mathbb{Z}$. Then **1**-connex is generated by $\mathrm{e}^{\mathrm{i}\,J(f)}$, with selfadjoint smeared fields "$J(f) := \sum_{a,n} f_n^a J_n^a$"

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Representations for super-current algebra net (sketch)

- Any irrep $(\pi, \mathcal{H})$ as above extends to the super-Virasoro algebra via ▸ Sugawara construction for L_n^π, G_r^π, with positive energy.
- Finitely many unitary irreducible representations π_q, classified by the value $J_0^{a,\pi} = q_a \mathbf{1}$, with $q \in R_N$ (cf. Kac): example

- Explicit localised endomorphisms constructed, acting trivially on the fermionic part. ▸ Details

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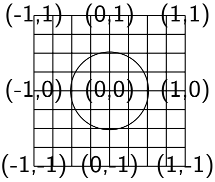
- 

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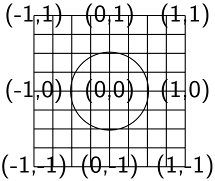
- 

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Ad (1): Spectral triple nets

Definition

A *net of θ-summable (graded) spectral triples* $(\mathfrak{A}(I), \mathcal{H}, Q)_{I\in\mathcal{I}}$ over $\mathbb{S}$ consists of

- (graded) Hilbert space $\mathcal{H}$;
- odd selfadjoint operator Q on $\mathcal{H}$ with associated superderivation $(\delta, \mathrm{dom}(\delta))$, and s.t. e^{-tQ^2} trace-class, for $t \in \mathbb{R}_+$;
- isotone net $\mathfrak{A}$ of unital graded *-algebras with (graded) representation $(\pi, \mathcal{H})$ such that $\pi_I(\mathfrak{A}(I)) \subset \mathrm{dom}(\delta)$.

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Ad (2): Entire Cyclic Cohomology and JLO Cocycle

Suitable generalisation of de Rham cohomology to noncommutative geometry is (entire) cyclic homology:

Definition

Letting $(A, ||\cdot||)$ be a Banach algebra, its *entire cyclic complex* $(EC_*(A), \partial) = (EC_e(A) \oplus EC_o(A), \partial)$ is given by chains $a = (a^{(n)})_{n\in\mathbb{N}_0} \in \bigoplus_{n\in\mathbb{N}_0} A^{\otimes(2n+1)} \oplus \bigoplus_{n\in\mathbb{Z}} A^{(\otimes(2n+2)}$ satisfying the entireness growth condition

$$n \in \mathbb{N} \mapsto \sqrt{n!}||a^{(n)}||, \quad \text{bounded},$$

and with boundary operator $\partial := b + B$. ▸ Details

Ad (2): Entire Cyclic Cohomology and JLO Cocycle

Theorem

(1) *Given a θ-summable graded spectral triple $((A,\gamma),(\pi,\mathcal{H},\Gamma),Q)$, consider Sobolev norm*

$$||\cdot||_* := ||\cdot|| + ||\delta(\pi(\cdot))||.$$

$$\tau^{(n)}(a_0,...a_n) := \int_{0\le t_1\le\ldots\le t_n\le 1} \operatorname{tr}\Big(\Gamma^{n+1}\pi(a_0)\,\mathrm{e}^{-t_1Q^2}[Q,\pi(a_1)]\,\mathrm{e}^{-(t_2-t_1)Q^2}\ldots$$
$$\ldots[Q,\pi(a_n)]\,\mathrm{e}^{-(1-t_n)Q^2}\Big)\,\mathrm{d}^{(n)}t.$$

Then JLO cocycles $(\tau^{(n)})_{n\in2\mathbb{N}_0}\oplus(\tau^{(n)})_{n\in2\mathbb{N}_0+1}\in C^*(A^\gamma)$.

(2) *If $(Q_t)_{t\in[0,1]}$ is a norm-continuous homotopy, then $\tau^t\simeq\tau^s$.*

(3) *For $p,u\in \mathrm{M}_r(A^\gamma)$, associate certain ECC's $\mathrm{Ch}_e(p)$ and $\mathrm{Ch}_o(u)$.*

(4) *Well-defined pairing with K-theory. Graded case:*

$$[p]\in K_0(A^\gamma)\mapsto \tau(\mathrm{Ch}_e(p)) = \operatorname{ind}_{\pi(p)\bar{\mathcal{H}}_+}(\pi(p)\bar{Q}\pi(p))_+ \in\mathbb{Z}.$$

Ungraded case:

$$[u]\in K_1(A^\gamma)\mapsto \tau(\mathrm{Ch}_o(u)) = \mathrm{sf}(\bar{Q},\pi(u)\bar{Q}\pi(u)^*)\in\mathbb{Z}$$

Model Analysis: Super-Current Algebra Models

Superderivation and its domain

Definition

A graded representation $(\pi, \mathcal{H})$ is called *supersymmetric* if it admits an odd square-root Q of L_0 on $\mathcal{H}$.

Facts:

- Supersymmetric iff π Ramond rep. Precise statements
- Then $Q := G_0^\pi$ (odd s.a. summable), associated superderivation $(\delta, \mathrm{dom}(\delta))$; moreover dense local domains $\mathrm{dom}(\delta) \cap \pi(\mathcal{A}(I)) \subset \pi(\mathcal{A}(I))$.
- Set $\mathfrak{A} := \pi^{-1}(\mathrm{dom}(\delta)) \subset C^*_{\mathrm{univ}}(\mathcal{A}^\gamma)$ and $\mathfrak{A}(I) := \mathfrak{A} \cap \mathcal{A}(I)$.

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The spectral triples

$\mathfrak{A} := \pi^{-1}(\mathrm{dom}(\delta)) \subset C^*_{\mathrm{univ}}(\mathcal{A}^\gamma)$ is normed algebra with suitable norm (taking account of δ and ρ), invariant under all $\rho \in \Delta$, thus we have

1. nontrivial global spectral triple $(\mathfrak{A}, \mathcal{H}, Q)$ and net of spectral triples $(\mathfrak{A} \cap \mathcal{A}(I), \mathcal{H}, Q)_{I \in \mathcal{I}}$,
2. well-defined JLO cocycle τ on $\mathfrak{A}$ and pullbacks $\rho^*\tau$, for $\rho \in \Delta$.

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Even index pairing (for graded rep's)

- recall

$$[p] \in K_0(A^\gamma) \mapsto \tau(\mathrm{Ch}_e(p)) = \mathrm{ind}_{\pi(p)\mathcal{H}_+}(\pi(p)Q\pi(p))_+ \in \mathbb{Z}$$

- define

$$p_{q,+} := \chi_1(e^{-(L_0 - \frac{q^2}{2})})\chi_1(e^{-||J_0 - q||^2})\frac{(1+\Gamma_0)}{2} = \rho_q^{-1}(p_{0,+})$$

what's the meaning of these three terms?

Theorem

The pull-back entire cyclic cocycles $\rho^*\tau$ over $\mathfrak{A}$, with $\rho \in \Delta$ are well-defined, the projections $p_{l,+}$ lie in $\mathfrak{A}$, and

$$\rho^*_{q_2,l}\tau(p_{q_1,+}) \sim \delta_{q_1,q_2}.$$

Hence, the cocycles $\rho^*\tau$, depending only on the sector $[\rho] \in [\Delta]$, can be completely separated by the family $\{p_{q,+} : q \in R_N\}$. In particular, they are mutually inequivalent.

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Odd index pairing (for ungraded rep's)

- recall
$$[u] \in K_1(A^\gamma) \mapsto \tau(\mathrm{Ch}_o(u)) = \mathrm{sf}(Q, \pi(u)Q\pi(u)^*) \in \mathbb{Z}$$
- define
$$u_{0,+} := p_{0,+}\gamma_d + (\mathbf{1} - p_{0,+})\mathbf{1}, \quad u_{q,+} := \rho_q^{-1}(u_{0,+}) \in C^*_{\mathrm{univ}}(\mathcal{A}^\gamma), \quad q \in R_N$$
why? what's the meaning of the terms here? well-definedness?

Theorem

The pull-back cocycles $\rho^*_{q,I}\tau$ are well-defined, the unitaries $u_{q,+}$ lie in $\mathfrak{A}$, and

$$\rho^*_{q_2,I}\tau(u_{q_1,+}) = \mathrm{sf}(Q, \pi\rho_{q_2,I}(u_{q_1,+})^*\, Q\, \pi\rho_{q_2,I}(u_{q_1,+})) = \dim(\mathcal{H}_{0,+})\delta_{q_1,q_2}.$$

Hence, the cocycles $\rho^*\tau$, depending only on the sector $[\rho] \in [\Delta]$, can be completely separated by the family $\{u_{q,+} : q \in R_N\}$. In particular, they are mutually inequivalent.

Conclusions for this model

- suitable family Δ of localised endomorphisms ρ_q of $\mathcal{A}^\gamma$ and Δ-invariant subalgebra $\mathfrak{A} \subset C^*_{\text{univ}}(\mathcal{A}^\gamma)$ established
- associated spectral triple $(\mathfrak{A}, \mathcal{H}, Q)$ and JLO cocycle $\tau \in EC^*(\mathfrak{A})$
- even/odd pairing for d even/odd: family in $K_{0,1}(\mathfrak{A})$ separating the $\rho\tau_{e,o} \in EC^{e,o}(\mathfrak{A})$, for $\rho \in \Delta$
- odd (ungraded) pairing for d odd: family in $K_1(\mathfrak{A})$ separating the $\rho\tau_o \in EC^o(\mathfrak{A})$, for $\rho \in \Delta$

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The General Index Pairing

The recipe

Given a superconformal net $\mathcal{A}$ with a supersymmetric Ramond representation $(\pi, \mathcal{H}, \Gamma)$ and supercharge $Q = G_0^\pi$.

Definition

A $\mathrm{PSL}(2,\mathbb{R})$-covariant locally normal positive energy endomorphism ρ of $\mathcal{A}^\gamma$ localised in a certain interval $I \in \mathcal{I}$ is called *differentiably transportable* if

$$\pi(z(\rho, g)) \in \mathrm{dom}(\delta), \quad g \in \mathrm{PSL}(2,\mathbb{R})^{(\infty)} \text{ small.}$$

The set of such ρ is denoted by Δ_I, and $\Delta := \bigcup_{I \in \mathcal{I}} \Delta_I$.

Definition

The Δ-*invariant differentiable field algebra* for $\mathcal{A}^\gamma$ is given by

$$\mathfrak{A}_\Delta := \{x \in \mathrm{vN}(\mathcal{A}^\gamma) : (\forall \rho \in \Delta)\ \pi \circ \rho(x) \in \mathrm{dom}(\delta)\}$$

and $\mathfrak{A}_\Delta(I) := \mathfrak{A}_\Delta \cap \mathcal{A}(I)$, and it becomes a Banach algebra with norm

$$||\cdot||_{*,\Delta} := ||\cdot|| + \sum_{[\rho] \in [\Delta]} ||\delta(\pi \circ \rho(\cdot))||.$$

The recipe

Given a superconformal net $\mathcal{A}$ with a supersymmetric Ramond representation $(\pi, \mathcal{H}, \Gamma)$ and supercharge $Q = G_0^\pi$.

Definition

A $\mathrm{PSL}(2,\mathbb{R})$-covariant locally normal positive energy endomorphism ρ of $\mathcal{A}^\gamma$ localised in a certain interval $I \in \mathcal{I}$ is called *differentiably transportable* if

$$\pi(z(\rho, g)) \in \mathrm{dom}(\delta), \quad g \in \mathrm{PSL}(2,\mathbb{R})^{(\infty)} \text{ small.}$$

The set of such ρ is denoted by Δ_I, and $\Delta := \bigcup_{I \in \mathcal{I}} \Delta_I$.

Definition

The Δ-*invariant differentiable field algebra* for $\mathcal{A}^\gamma$ is given by

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Theorem

- *If* $\mathcal{A}^\gamma$ *satisfies the split property and* $\mathrm{dom}(\delta) \cap \pi(\mathcal{A}^\gamma(I))$ *is non-trivial, then:*
  - *The set of diff.-transportable endo's* Δ *forms a local tensor* C^**-category,*
  - $(\mathfrak{A}_\Delta, \mathcal{H}, Q)$ *provides a non-trivial* θ*-summable spectral triple, whose endomorphisms include* Δ*, and whose JLO cocycle* τ *on* $\mathfrak{A}_\Delta$ *is well-defined.*
- *Suppose* $\mathcal{H}$ *is graded. With some* assumptions *on the lowest energy space and the existence of a certain projection* $p_0 \in \mathfrak{A}$ *with nonvanishing index, the family* $\{p_\sigma := \sigma^{-1}(p_0) : [\sigma] \in [\Delta]\}$ *separates all sectors:*

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Conclusions

- Novel procedure: graded-local conformal net $\rightarrow$ suitable differentiably transportable endomorphisms $\rightarrow$ (net of) θ-summable even/odd spectral triples $\rightarrow$ even/odd entire cyclic JLO cocycles and pullback $\rightarrow$ even/odd index pairing and explicit evaluation.
- Modest assumptions. Fulfilled by several models.
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Suppose that either

- $\mathrm{ind}_{p_{\mathrm{lw}\,\pi_R}\mathcal{H}_+}(p_{\mathrm{lw}\,\pi_R} Q p_{\mathrm{lw}\,\pi_R})_+ \neq 0$ in which case $p_\sigma := \sigma^{-1}(p_{\mathrm{lw}\,\pi_R})$, or
- there is $e \in \mathcal{A}$ such that $ep_{\mathrm{lw}\,\pi_R} \in \mathfrak{A}_\Delta$ is a projection and

$$\mathrm{ind}_{ep_{\mathrm{lw}\,\pi_R}\mathcal{H}}(ep_{\mathrm{lw}\,\pi_R} Q ep_{\mathrm{lw}\,\pi_R}) \neq 0,$$

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- there is a family $\{e_\sigma : \sigma \in \Delta\}$ such that $e_\sigma \sigma^{-1}(p_{\mathrm{lw}\,\pi_R}) \in \mathfrak{A}_\Delta$ is a projection and $\mathrm{ind}_{\sigma(e_\sigma)p_{\mathrm{lw}\,\pi_R}\mathcal{H}}(\sigma(e_\sigma)p_{\mathrm{lw}\,\pi_R} Q \sigma(e_\sigma)p_{\mathrm{lw}\,\pi_R}) \neq 0$, in which case $p_\sigma := e_\sigma \sigma^{-1}(p_{\mathrm{lw}\,\pi_R})$.

If π_R is the only representation of $\mathcal{A}$ with lowest energy $\mathrm{lw}\,\pi_R$, then the family $\{p_\sigma : \sigma \in \Delta\}$ completely distinguishes all sectors in the sense that

$$\rho^*\tau(p_\sigma) = \delta_{[\sigma],[\rho]}\,\mathrm{ind}_{p_{\mathrm{lw}\,\pi_R}\mathcal{H}}(p_{\mathrm{lw}\,\pi_R} Q p_{\mathrm{lw}\,\pi_R}), \quad \sigma,\rho \in \Delta.$$

If there is more than one sector with lowest energy $\mathrm{lw}\,\pi_R$, we have at least

Return

$$\rho^*\tau(\sigma^{-1}(e_{\mathrm{lw}\,\pi_R} p_{\mathrm{lw}\,\pi_R})) = \left\{ \begin{array}{ll} 0 & : \mathrm{lw}(\sigma) \neq \mathrm{lw}(\rho) \\ \mathrm{ind}_{p_\iota \mathcal{H}_+}(p_\iota Q p_\iota)_+ & : [\sigma] = [\rho] \\ \text{undetermined} & : \text{otherwise} \end{array} \right\} \quad \sigma,\rho \in \Delta.$$

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$$L_n^{\pi} = \frac{1}{2}\sum_a \sum_m \left(: J_{n-m}^{a,\pi} J_m^{a,\pi} : + m : F_{n-m}^{a,\pi} F_m^{a,\pi} :\right) + \frac{d}{16}\delta_{n,0}$$

$$G_r^{\pi} = \sum_a \sum_s J_{r-s}^{a,\pi} F_s^{a,\pi}$$

Superderivation and its domain

Definition

A graded representation $(\pi, \mathcal{H})$ is called *supersymmetric* if it admits an odd square-root Q of L_0 on $\mathcal{H}$.

Proposition

- *Neveu-Schwarz representations of $\mathcal{A}$ are never supersymmetric, but Ramond representations are so.*
- *Neveu-Schwarz (Ramond) unitary irreducible representations of $\mathcal{A}$ correspond one-to-one to Neveu-Schwarz (Ramond) unitary irreducible representations of the super-current algebra.*

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Theorem

Let $(\pi, \mathcal{H})$ be the lowest irreducible Ramond representation of $\mathcal{A}$, and $Q = G_0^\pi$ a supercharge with superderivation δ on $B(\mathcal{H})$. Then

$$*alg\{\pi_I(\Psi(f,j)), \quad \pi_I(F(f)(J(f)+\mathrm{i})^{-1}) : \ f \in C_I^\infty(\mathbb{S}, \mathbb{R}^d), j \in L_N \ etc\} \subset \mathrm{dom}(\delta).$$

Generated von Neumann algebra coincides with $\pi_I(\mathcal{A}(I))$, so $\pi_I^{-1}(\mathrm{dom}(\delta)) \cap \mathcal{A}(I) \subset \mathcal{A}(I)$ weakly dense. Define

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Construct smooth localised automorphisms of $\mathcal{A}$: considering a differentiable function $\iota_I : \mathbb{R} \to \mathbb{R}$ such that $\iota_I(t + 2\pi) = \iota_I(t) + 2\pi$ and $\iota_{I|(\mathbb{S}\setminus I)} = 0$, then

$$\rho_{q,I} : \mathrm{e}^{\mathrm{i}\, J(\tilde{f})}\, \gamma_j \mapsto \mathrm{e}^{\mathrm{i} \int \iota_I' \langle q, \tilde{f} \rangle}\, \mathrm{e}^{\mathrm{i}\, J(\tilde{f})}\, \gamma_j, \quad F(f) \mapsto F(f), \quad q \in R_N$$

Return

Ad (2): Entire Cyclic Cohomology and JLO Cocycle

Return Suitable generalisation of de Rham cohomology to noncommutative geometry is (entire) cyclic homology:

Definition

Letting $(A, ||\cdot||)$ be a Banach algebra, its *entire cyclic complex* $(EC_*(A), \partial) = (EC_e(A) \oplus EC_o(A, \partial)$ is given by chains $a = (a^{(n)})_{n\in\mathbb{N}_0} \in \bigoplus_{n\in\mathbb{N}_0} A^{\otimes(2n+1)} \oplus \bigoplus_{n\in\mathbb{Z}} A^{(\otimes(2n+2)}$ satisfying the entireness condition

$$n \in \mathbb{N} \mapsto \sqrt{n!}||a^{(n)}||, \quad \text{bounded},$$

and with boundary operator $\partial := b + B$:

$$\begin{aligned} b(a_0 \otimes \ldots \otimes a_n) &:= \sum_{k=0}^{n-1} (-1)^k (a_0 \otimes \ldots \otimes (a_k a_{k+1}) \otimes \ldots \otimes a_n) \\ &\quad + (-1)^n ((a_n a_0) \otimes a_1 \otimes \ldots \otimes a_{n-1}), \\ B(a_0 \otimes \ldots \otimes a_n) &:= \sum_{k=0}^{n} (-1)^{nk} (\mathbf{1} \otimes a_k \otimes \ldots \otimes a_{k-1}). \end{aligned}$$

Entire cyclic cocomplex defined dually. Induced (co-)homology.