

Spectral triples for the Sierpinski gasket

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work in progress with F. Cipriani, T. Isola, J-L. Sauvageot

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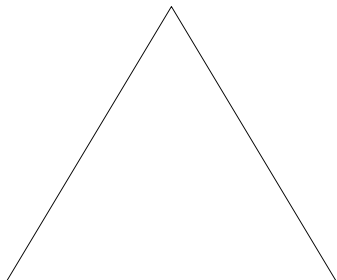
Outline

- 1 Introduction
- 2 Triples on Fractals - Some examples
- 3 Further triples on S^1
- 4 Deformed triples for the gasket
- 5 An energy for the gasket

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The Sierpinski gasket



Sierpinski: $K = \bigcup_{i=1,2,3} w_i(K)$

Similarities: $w_\sigma = w_{\sigma_n} \cdot \dots \cdot w_{\sigma_1}$, $|\sigma| = n$.

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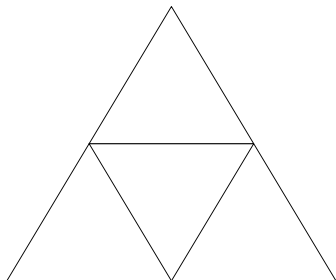
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$\sigma(e)$ = origin of e , $t(e)$ = terminus of e

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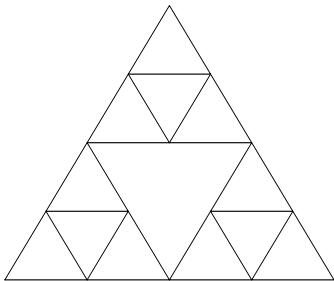
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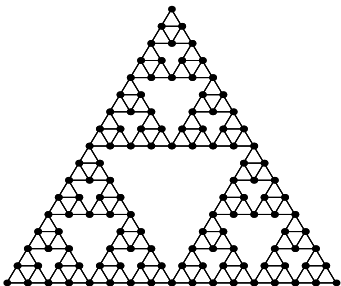
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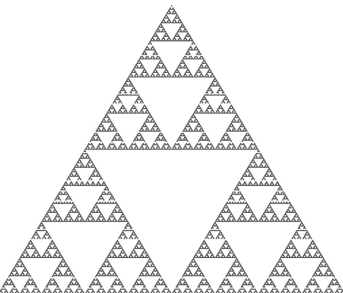
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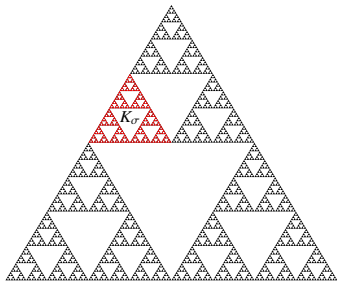
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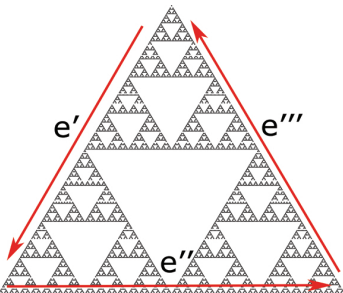
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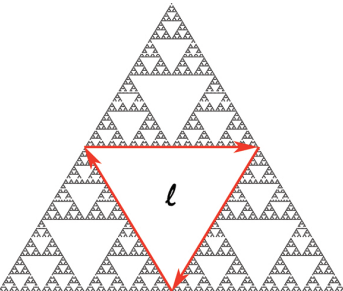
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Direct sum triples

Natural choice: define a triple as a direct sum of triples associated with suitable building blocks.

Goals:

- (Hausdorff) measure and dimension
- (geodesic) distance
- non trivial pairing with K-theory
- (Kigami) energy

Measure and dimension for a triple

Spectral triple (on a C^* -algebra \mathcal{A}): (π, \mathcal{H}, D) , where \mathcal{H} is a separable Hilbert space, π is a representation of \mathcal{A} on \mathcal{H} , D an (unbounded) selfadjoint operator acting on \mathcal{H} such that:

- D has compact resolvent,
- $[D, a]$ is bounded for a in a dense subalgebra of \mathcal{A} .

Integration: $\int f \longleftrightarrow \text{tr}_\omega a |D|^{-d}$, where:

$$\text{tr}_\omega T = \lim_\omega \frac{\sum_1^n \mu_k(T)}{\log n}, \quad T > 0 \text{ (logarithmic Dixmier trace)}$$

$$d = \inf\{s : \sup_n \frac{\sum_1^n \mu_k(|D|^{-s})}{\log n} < \infty\} \text{ (metric dimension).}$$

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Distance and Energy

Noncommutative distance is generally defined between states:

$$d(x, y) = \sup_f \frac{|f(x) - f(y)|}{Lip(f)} \longleftrightarrow d(\varphi, \psi) = \sup_a \frac{|\varphi(a) - \psi(a)|}{\|[D, a]\|}$$

The energy of an element can be defined as:

$$\mathcal{E}[f] = \int |\nabla f|^2 \longleftrightarrow \mathcal{E}[a] = \text{tr}_\omega \|[D, a]\|^2 |D|^{-d}$$

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Pairing with K-Theory - Fredholm modules

Spectral triple on $\mathcal{A} \longrightarrow$ Fredholm module on \mathcal{A} ,

$$(\pi, \mathcal{H}, D) \longrightarrow (\pi, \mathcal{H}, F),$$

where F is the phase of D .

In the odd case, for $u \in \mathcal{M}_k \otimes \mathcal{A}$, the pairing is given by:

$$\varphi([u]) = \text{Index}(P_+^k u P_+^k),$$

where P_+ is the spectral projection of F for the eigenvalue 1, and $P_+^k = I_k \otimes P_+$.

Multigraded Fredholm modules

A p -multigraded Fredholm module is an even Fredholm module $(\pi, \mathcal{H}, F, \gamma)$ with p odd skew-adjoint unitaries, anticommuting with one another, and commuting with F and $\pi(\mathcal{A})$. If $p = 0$ we get an even Fredholm module, if $p = -1$ we get an odd Fredholm module. They pair with K -theory according to the parity of p . For example, a 1-multigraded Fredholm module $(\pi, \mathcal{H}, F, \gamma, \varepsilon)$ pairs with $K_1(\mathcal{A})$ as

$$\varphi([u]) = \text{Index}(\tilde{P}_+^k u \tilde{P}_+^k),$$

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Cantor set (Connes '94)

$$\ell = \left\{ \frac{1}{3}, \frac{2}{3} \right\}, \ell_\sigma = w_\sigma \ell, \sigma \in \bigcup_n \{1, 2\}^n, \mathcal{A} = \mathcal{C}(\text{Cantor}),$$

$$(\pi, \mathcal{H}, D) = \bigoplus_{\sigma} (\pi_\sigma, \mathcal{H}_\sigma, D_\sigma), \mathcal{H}_\sigma = \ell^2(\ell_\sigma),$$

$$\pi_\sigma(f) = \begin{pmatrix} f(w_\sigma(\frac{1}{3})) & 0 \\ 0 & f(w_\sigma(\frac{2}{3})) \end{pmatrix}, D_\sigma = 3^{-|\sigma|} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ (even triple).}$$

Theorem

$$\text{tr}_\omega(f|D|^{-d}) = c \int f dH_d, d = \frac{\log 2}{\log 3}; \sup_f \frac{|f(x)-f(y)|}{\|[D, f]\|} = d(x, y);$$

$$\varphi([P_\sigma]) = \text{Index} \left(\begin{pmatrix} 0 & 0 \\ 0 & P_\sigma \end{pmatrix} F \begin{pmatrix} P_\sigma & 0 \\ 0 & 0 \end{pmatrix} \right) = 1, P_\sigma = \chi_{[0, w_\sigma(1/3)]};$$

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First attempts for the gasket

$$\mathcal{A} = \mathcal{C}(K), (\pi, \mathcal{H}, D) = \bigoplus_{e \in E} (\pi_e, \mathcal{H}_e, D_e), E \text{ edges.}$$

$$\text{D.G.-Isola('03): } \mathcal{H}_e = \ell^2\{o(e), t(e)\}, D_e = \frac{1}{|e|} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Christensen-Ivan-Lapidus('08): $\mathcal{H}_e = L^2(e)$, D_e derivation.

In both cases measure, dimension and distance are recovered, but trivial pairing with K-theory.

In the first case, Kigami energy may be reconstructed.

Lacunary triples for the gasket K

$$\mathcal{A} = \mathcal{C}(K), (\pi, \mathcal{H}, D) = \bigoplus_{\sigma} (\pi_{\sigma}, \mathcal{H}_{\sigma}, D_{\sigma}), \sigma \in \bigcup_n \{1, 2, 3\}^n.$$

Theorem (Christensen, Ivan, Schrohe '10)

If $\mathcal{H}_{\emptyset} = L^2(\ell_{\emptyset} \equiv S^1)$, $\pi_{\emptyset}(f) = f|_{\ell_{\emptyset} \equiv S^1}$, $D_{\emptyset} = i \frac{d}{d\vartheta}$, $\mathcal{H}_{\sigma} = \mathcal{H}_{\emptyset}$,

$\pi_{\sigma}(f) = \pi_{\emptyset}(f \cdot w_{\sigma})$, $D_{\sigma} = 2^{-|\sigma|} D_{\emptyset}$, then:

$\text{tr}_{\omega}(f|D|^{-d}) = \text{const} \int f dH_d$, $d = \frac{\log 3}{\log 2}$;

$\sup_f \frac{|f(x) - f(y)|}{\|[D, f]\|} = d_{\text{geo}}(x, y)$ (up to minor modifications);

$\varphi([u]) = \text{Index}(P_+ u P_+) = \sum_{\sigma} c_{\sigma}(u)$, where P_+ is the spectral projection on the positive part of D , u is a unitary in \mathcal{A} and $c_{\sigma}(u)$ is the winding number of u w.r.t. ℓ_{σ} .

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1-multigraded triples on S^1

$$\mathcal{H} = L^2(\Omega(S^1)) = L^2(\Omega^1(S^1)) \oplus L^2(\Omega^0(S^1)), \quad \pi(f) = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix},$$

$D = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In order to recover the pairing with odd K-theory, we need a further unitary, odd, skewadjoint ε such that $[\varepsilon, \pi(f)] = 0, [\varepsilon, F] = 0$, thus getting a 1-multigraded Fredholm module ($F = \text{phase of } D$). We set $\varepsilon = \begin{pmatrix} 0 & -iU \\ -iU^* & 0 \end{pmatrix}$,

with $Uf = f d\vartheta$. Then, setting $\tilde{F} = i\varepsilon F$, $(\pi, \mathcal{H}, \tilde{F})$ is an odd Fredholm module, and the pairing with odd K-theory is given by $\varphi([e_k]) = \text{Index}(P_+ e_k P_+)$, where P_+ is the spectral projection of \tilde{F} for the eigenvalue 1, and $e_k = e^{2\pi i k \vartheta}$.

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Deformed 1-multigraded triples on S^1

For any $\alpha \in (0, 1)$, we set $\mathcal{H} = L^2(S^1 \times S^1) \oplus L^2(S^1)$,
 $\pi(f) = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$, $D = \begin{pmatrix} 0 & \partial_\alpha \\ \partial_\alpha^* & 0 \end{pmatrix}$, $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where
 $(\partial_\alpha f)(x, y) = \frac{f(x) - f(y)}{N(x-y)}$, $N(t) \sim |t|^{(1/2+\alpha)}$.

Observations:

- $\|f\|_\alpha := \|\partial f\|$ is a norm for the Sobolev space H^α ,
- setting $e_k = e^{2\pi i k \vartheta}$, we have $\partial^* \partial e_k = \lambda_k e_k$; we choose $N(t)$ s.t. $\lambda_k = |k|^{2\alpha}$.
- $\dim(\ker(D)) = \infty$, so D has no compact resolvent. It will cause no problems in the following.
- undeformed triple corr. to $\alpha = 1$

Partial multigrading

The analog of the ε above is given by $\varepsilon = \begin{pmatrix} 0 & -iU \\ -iU^* & 0 \end{pmatrix}$,

where $Ue_k = \text{sgn}(k)\lambda_k^{-1/2}\partial e_k$. However, in this case, $[\varepsilon, \pi(f)] \neq 0$; we call such an ε a partial multigrading.

Proposition

Setting $\tilde{F} = i\varepsilon F$, we get an odd Fredholm module.

$$\varphi([e_k]) = \text{Index}(P_+ e_k P_+) = 2k,$$

where P_+ is the spectral projection of \tilde{F} for the eigenvalue 1.

NB: the compactness of $[\tilde{F}, \pi(f)]$ has to be proven directly.

Features of the α -deformed triples on S^1

Proposition

$\text{tr}_\omega(f|D|^{-1/\alpha}) = c \int f(x) dx$. Let $d_\alpha(x, y) = \sup_f \frac{|f(x) - f(y)|}{\|[D, f]\|}$. Then

$$c_1 \left(\log \frac{1}{2d(x, y)} \right)^{-1/2} d(x, y)^\alpha \leq d_\alpha(x, y) \leq c_2 d(x, y)^\alpha,$$

$$\mathcal{E}_\alpha[f] = \text{tr}_\omega(\|[D, f]\|^2 |D|^{-1/\alpha}) = \text{const} (f, \Delta^\alpha f).$$

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A family of triples for the gasket

Let $c > 1$, $\alpha \in (0, 1]$, $\mathcal{A} = \mathcal{C}(K)$, $\ell_\emptyset \equiv S^1$ isometrically.

Set $A_{c,\alpha} = (\pi, \mathcal{H}, D) = \bigoplus_{\sigma} (\pi_{\sigma}, \mathcal{H}_{\sigma}, D_{\sigma})$, where

$$\mathcal{H}_{\emptyset} = L^2(\ell_{\emptyset} \times \ell_{\emptyset}) \oplus L^2(\ell_{\emptyset}), \quad \pi_{\emptyset}(f) = \begin{pmatrix} f|_{\ell_{\emptyset}} & 0 \\ 0 & f|_{\ell_{\emptyset}} \end{pmatrix}, \quad D_{\emptyset} = \begin{pmatrix} 0 & \partial \\ \partial^* & 0 \end{pmatrix},$$

and $\partial = \partial_{\alpha}$ as above.

Then set $\mathcal{H}_{\sigma} = \mathcal{H}_{\emptyset}$, $\pi_{\sigma}(f) = \pi_{\emptyset}(f \cdot w_{\sigma})$, $D_{\sigma} = c^{|\sigma|} D_{\emptyset}$.

Dixmier trace and Hausdorff measure

Theorem

$$\mathcal{Z}_D(s) = \text{tr}(|D|^{-s}) = \frac{4\zeta(\alpha s)}{1 - 3c^{-s}}, \quad \zeta \text{ being the Riemann zeta.}$$

$$\text{Dimensional spectrum} = \{\alpha^{-1}\} \cup \left\{ \frac{\log 3}{\log c} \left(1 + \frac{2\pi i}{\log 3} k \right) : k \in \mathbb{Z} \right\}.$$

When $1 < c < 3^\alpha$, i.e. $d_D = \frac{\log 3}{\log c}$,

$$\text{tr}_\omega(f|D|^{-d}) = \frac{4d}{\log 3} \frac{\zeta(d)}{(2\pi)^d} \int_K f dH_\gamma, \quad \text{with } \gamma = \frac{\log 3}{\log 2}.$$

In particular, for $c = 2$ (and $\alpha > \frac{\log 2}{\log 3}$), Hausdorff measure and dimension are recovered.

N.B.: As usual, $|D|^{-s}$ is meant to be 0 on $\ker D$.

The commutator condition

Theorem

Let us consider the triple $A_{c,\alpha}$ on the gasket. If $c \leq 2$ and $f \in C^{0,1}$, then

$$\|[D, f]\| \leq \text{const} \|f\|_{C^{0,1}}.$$

As a consequence, for any $c \leq 2$, $A_{c,\alpha}$ is a spectral triple.

N.B.: the estimate above is optimal only if $c = 2$. If $c < 2$, we have $\|[D, f]\| \leq \text{const} \|f\|_{C^{0,\beta}}$, with $\beta \geq \frac{\log c}{\log 2}$, $\beta > \alpha$.

Distance on the gasket for $c = 2$

Theorem

Assume $c = 2$, and denote by d_D the noncommutative distance induced by the Dirac operator, and by d_{geo} the geodesic distance on the gasket induced by the euclidean metric on the plane. Then,

$$const_1 d_{geo}(x, y) \leq d_D(x, y) \leq const_2 d_{geo}(x, y).$$

N.B.: even though the α -deformed triple on S^1 induces a distance not too far from $d(x, y)^\alpha$, the triple $A_{2,\alpha}$ induces distance which is bi-Lipschitz w.r.t. the (Euclidean) geodesic distance.

Pairing with K-theory

Let us denote by F the phase of D , and add a partial multigrading to the triple above: $\varepsilon = \bigoplus_{\sigma} \varepsilon_{\sigma}$, where ε_{σ} is the one described above for the α -deformed triple on S^1 .

Theorem

*Setting $\tilde{F} = i_{\varepsilon} F$, we get an odd Fredholm module on $\mathcal{C}(K)$.
 $\varphi([u]) = \text{Index}(P_+ u P_+) = 2 \sum_{\sigma} c_{\sigma}(u)$, where P_+ is the spectral projection of \tilde{F} for the eigenvalue 1, $u \in \mathcal{C}(K)$ is a unitary, and $c_{\sigma}(u)$ is the winding number of u w.r.t. ℓ_{σ} .*

N.B.: the unitaries $u_{\sigma} : c_{\rho}(u_{\sigma}) = \delta_{\rho\sigma}$ generate $K_1(\mathcal{C}(K))$.

Outline

- 1 Introduction
- 2 Triples on Fractals - Some examples
- 3 Further triples on S^1
- 4 Deformed triples for the gasket
- 5 An energy for the gasket

A formula for spectral triples

To define the energy, we replace $\int |\nabla f|^2$ with $\text{tr}_\omega(|[D, f]|^2 |D|^{-\delta})$.

Indeed, for many fractals, energy measures are singular w.r.t. volume measures, hence we expect the spectral dimension to change. Therefore, we look for the abscissa of convergence of

$$\text{tr}(|[D, f]|^2 |D|^{-s}) = \sum_{\sigma} c^{(2-s)|\sigma|} \text{tr}(|[D_{\emptyset}, \pi_{\emptyset}(f \cdot w_{\sigma})]|^2 |D_{\emptyset}|^{-s}).$$

Lemma

$\text{tr}(|[D_{\emptyset}, \pi_{\emptyset}(g)]|^2 |D_{\emptyset}|^{-s})$ is finite if and only if $g \in H^{\alpha}(\ell_{\emptyset})$.

Jonsson Theorem

Kigami energy on the gasket:

$$\mathcal{E}_n[f] = \left(\frac{5}{3}\right)^n \sum_{e \in E_n} |f(t(e)) - f(o(e))|^2, \quad \mathcal{E}[f] = \lim_n \mathcal{E}_n[f].$$

Theorem (Jonsson '05)

Let g be a function on the gasket with finite Kigami energy,

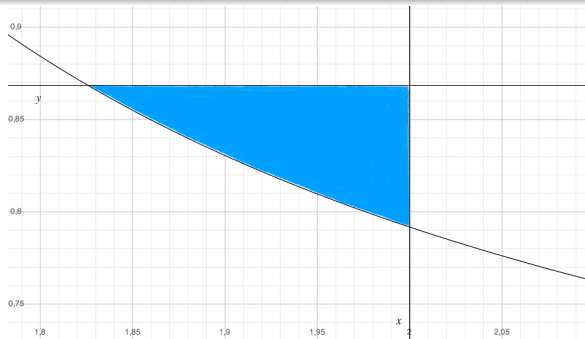
$\alpha_0 = \frac{\log 5}{\log 4} - \frac{1}{2} \left(\frac{\log 3}{\log 2} - 1 \right) \sim 0.87$. Then $g|_e \in H^{\alpha_0}(e)$ for any edge e . Conversely, given $\phi \in H^{\alpha_0}(e)$, it extends to a function g on K with finite Kigami energy.

Theorem

Assume that $\alpha \leq \alpha_0$, $c > (5/3)^{1/(2-\alpha^{-1})}$, $\mathcal{E}[f] < \infty$. Then the abscissa of convergence of the function $\text{tr}(|[D, f]|^2 |D|^{-s})$ is $\delta := 2 - \frac{\log 5/3}{\log c}$, where it has a simple pole. Therefore

$$\mathcal{E}_D[f] = \text{tr}_\omega(|[D, f]|^2 |D|^{-\delta}) = \text{Res}_{s=\delta} \text{tr}(|[D, f]|^2 |D|^{-s}) = \text{const } \mathcal{E}[f].$$

N.B.: $\delta < d$, namely $\text{tr}_\omega(|[D, f]|^2 |D|^{-d}) = 0$



$$x = c, y = \alpha.$$

For c, α in the region above, the spectral triple $A_{c, \alpha}$ gives: • a non trivial distance, • the Hausdorff measure, • a non trivial pairing with $K^1(K)$, • the Kigami energy up to a constant.
If $c = 2$ and $0.79 < \alpha \leq 0.85$, we get the correct Hausdorff measure and dimension, and the distance is bi-Lipschitz w.r.t. the geodesic distance.