Superconformal field theory and operator algebras

Yasu Kawahigashi

University of Tokyo

Cardiff

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Quantum field theory and von Neumann algebras

Outline of the talk:

1. $N = 1$ **Super Virasoro algebras and von Neumann algebras**
   (with Carpi, Longo)
2. $N = 1$ **Super Virasoro algebras and super Moonshine**
3. $N = 1$ **Super Virasoro algebras and noncommutative geometry**
   (with Carpi, Hillier, Longo) → Hillier's talk tomorrow
4. $N = 2$ **Super Virasoro algebras, superstring theory and von Neumann algebras**
   (with Carpi, Longo, Xu — in progress)

The Virasoro algebra is an infinite dimensional Lie algebra generated by $\{L_n \mid n \in \mathbb{Z}\}$ and a central element c with the following relations.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

The Lie group $\mathrm{Diff}(S^1)$ gives a Lie algebra generated by $L_n = -z^{n+1}\frac{\partial}{\partial z}$. The Virasoro algebra is a central extension of the complexification of this.

We can classify its irreducible unitary highest weight representations, where the central charge c is mapped to a positive scalar.

(Some formal similarity to the Temperley-Lieb algebra.)

Super version: $N = 1$ Super Virasoro algebras

The infinite dimensional super Lie algebras generated by central element c, even elements L_n, $n \in \mathbb{Z}$, and odd elements G_r, $r \in \mathbb{Z}$ or $r \in \mathbb{Z} + 1/2$, with the following relations:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\
[G_r, G_s] &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}.
\end{aligned}$$

Ramond algebra, if $r \in \mathbb{Z}$

Neveu-Schwarz algebra, if $r \in \mathbb{Z} + 1/2$

Fix a vacuum representation π of the $N=1$ super Virasoro algebra and simply write L_n for $\pi(L_n)$.
Consider $L(z)=\sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, the stress-energy tensor, and $\sum_{r\in\mathbb{Z}+1/2} G_r z^{-r-3/2}$, super stress-energy tensor.
These power series with $z\in\mathbb{C}$, $|z|=1$ give operator-valued distributions on S^1.
Fix an interval I and take a C^∞-function f with $\operatorname{supp} f\subset I$.
We have (unbounded) operators $\langle L, f\rangle$, $\langle G, f\rangle$.
$A(I)$: the von Neumann algebra generated by these operators with various f.

The $\mathbb{Z}_2$-grading of the super Lie algebra passes to the $\mathbb{Z}_2$-grading of the operator algebras.

Quantum Field Theory: (mathematical setting)

1. Spacetime (e.g., Minkowski space)
2. Symmetry group (e.g., Poincaré group)
3. Quantum fields (operator-valued distributions on the spacetime)

Conformal Field Theory

Two-dimensional Minkowski space $\{(x,t) \mid x,t \in \mathbb{R}\}$
$\rightarrow$ One of the light rays $x = \pm t$ compactified to S^1.
Orientation preserving diffeomorphism group $\mathrm{Diff}(S^1)$.

We have operator-valued distributions acting on a Hilbert space of states having a vacuum vector.

Operator algebraic axioms: (superconformal field theory)

Motivation: Operator-valued distributions $\{T\}$ on S^1.

Fix an interval $I \subset S^1$, consider $\langle T, f\rangle$ with $\mathrm{supp}\, f \subset I$.

$A(I)$: the von Neumann algebra generated by these (possibly unbounded) operators

1. $I_1 \subset I_2 \;\Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \;\Rightarrow [A(I_1), A(I_2)] = 0$. **(graded commutator)**
3. $\mathrm{Diff}(S^1)$**-covariance (conformal covariance)**
4. **Positive energy**
5. **Vacuum vector**

Such a family $\{A(I)\}$ is called a superconformal net.

Each $A(I)$ is usually an injective type III_1 factor (the Araki-Woods factor). The even part of the superconformal net gives a local conformal net.

Representation theory of local conformal nets (Doplicher-Haag-Roberts):
We have a braided tensor category. Each representation is given by an endomorphism and its dimension is given by the square root of the Jones index of the image.

K-Longo-Müger: Complete rationality characterizes finiteness of the number of the irreducible representations and their finite dimensionality. ($\longrightarrow$ modular tensor category)
($\sim$ finite depth subfactors)

Moonshine

Mysterious relations between finite simple groups and elliptic modular functions.

Classification of finite simple groups:

1. **Cyclic groups of prime order**
2. **Alternating groups of degree 5 or higher**
3. **16 series of Lie type groups (such as $PSL(n, \mathbb{F}_q)$)**
4. **26 sporadic groups (since Mathieu, 1861)**

Monster has the largest order among 26 sporadic finite simple groups. The order is

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

which is around 8×10^{53}.

The smallest dimension of Monster's non-trivial irreducible representation is 196883.

The following function, called j-function, has been classically studied.

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

For $q = \exp(2\pi i\tau)$, $\mathrm{Im}\ \tau > 0$, we have modular invariance property, $j(\tau) = j\left(\frac{a\tau+b}{c\tau+d}\right)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, and this is the only function satisfying this property and starting with q^{-1}, up to freedom of the constant term.

$$196884 = 196883 + 1 \quad \text{(McKay)}$$

Similar relations for other coefficients of the j-function and dimensions of irreducible representations of the Monster group.

Then Conway-Norton formulated the Moonshine conjecture roughly as follows, which has been now proved by Borcherds.

(1) We have a "natural" infinite dimensional graded vector space $V = \bigoplus_{n=0}^{\infty} V_n$ with $\dim V_n < \infty$ having some algebraic structure whose automorphism group is the Monster group.

(2) For any element g in the Monster, the power series $\sum_{n=0}^{\infty} (\mathrm{Tr}\ g|_{V_n}) q^{n-1}$ is a special function called a Hauptmodul for some discrete subgroup of $SL(2,\mathbb{R})$. When g is the identity element, we obtain the j-function minus 744.

Construction of Frenkel-Lepowsky-Meurman of the Moonshine VOA (vertex operator algebra) for part (1):

VOA is a family of algebraic operator-valued distributions on S^1 for conformal field theory, so it should "correspond" to a local conformal net.

As an operator algebraic counterpart of the Moonshine VOA, we construct an extension of the 48th tensor power of the Virasoro net with $c = 1/2$. Then an extension of a $\mathbb{Z}_2$-fixed point gives a local conformal net called the Moonshine net with the expected properties. (K-Longo)

1. Representation theory is trivial
2. The automorphism group is the Monster
3. Hauptmodul property (as in the Moonshine conjecture)

First Conway group Co_1:

The order is $2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$, which is around 4×10^{18}. This is one of the 26 sporadic finite simple groups, and also the automorphism group of the Leech lattice in dimension 24 divided by its center. The other Conway groups, Co_2 and Co_3, are subgroups of Co_1.

Duncan constructed a "super" analogue of the Moonshine VOA and showed that its automorphism group is Conway's group Co_1.

We now have its operator algebraic counterpart having $c = 12$, but in order to get Co_1, we have to consider the group of automorphisms fixing the super Virasoro subnet pointwise, rather than the entire automorphism group. (K)

Rudvalis group:

Duncan considered the Rudvalis group, which is one of the six pariah groups not involved in the Monster group. (That is, it is not a quotient of a subgroup of the Monster group.) The order is $2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$, which is around 10^{11}.

He constructed two super VOA's with automorphic actions of the Rudvalis group with certain Moonshine type properties on two-variable power series arising from elements of the Rudvalis group.

We have an operator algebraic counterpart for one of the two, based on our general theory of superconformal nets. It has $c = 28$, and the group of automorphisms fixing a (strange) subnet pointwise gives the Rudvalis group. (K)

Geometric aspects of local conformal nets

Classical geometry: Consider the Laplacian Δ on an n-dimensional compact oriented Riemannian manifold. The classical Weyl formula gives an asymptotic expansion

$$\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots),$$

as $t \to 0+$, where a_0 is the volume of the manifold, and if $n = 2$, then a_1 is (constant times) the Euler characteristic of the manifold.

So the coefficients in the asymptotic expansion have a geometric meaning. We look for their analogues in the setting of superconformal nets.

The conformal Hamiltonian L_0 of a local conformal net is the generator of the rotation group of S^1.

For a nice local conformal net, we have an expansion

$$\log \mathrm{Tr}(e^{-tL_0}) \sim \frac{1}{t}(a_0 + a_1 t + \cdots),$$

where a_0, a_1, a_2 are explicitly given. (K-Longo)

This gives an analogy of the Laplacian Δ of a manifold and the conformal Hamiltonian L_0 of a local conformal net.

The classical Dirac operator is a "square root" of the Laplacian. Among the $N = 1$ super Virasoro generators, G_0 is "square root" of L_0. This idea gives a realization of the Dirac operator in NCG for $N = 1$ superconformal nets.

[$\longrightarrow$ Hillier's talk tomorrow.]

$N=2$ super Virasoro algebra (Ramond/N-S for $a = 0, 1/2$)

Generated by central element c, even elements L_n and J_n, and odd elements $G^{\pm}_{n\pm a}$, $n \in \mathbb{Z}$, with the following.

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[J_m, J_n] &= \frac{c}{3} m \delta_{m+n,0} \\
[L_n, J_m] &= -mJ_{m+n}, \\
[L_n, G^{\pm}_{m\pm a}] &= \left(\frac{n}{2} - (m \pm a)\right) G^{\pm}_{m+n\pm a}, \\
[J_n, G^{\pm}_{m\pm a}] &= \pm G^{\pm}_{m+n\pm a}, \\
[G^{+}_{n+a}, G^{-}_{m-a}] &= 2L_{m+n} + (n - m + 2a)J_{n+m} + \\
&\quad \frac{c}{3}\left((n+a)^2 - \frac{1}{4}\right)\delta_{m+n,0}.
\end{aligned}
$$

It is known that an irreducible unitary representation maps c to a scalar in the set

$$\left\{ \frac{3m}{(m+2)} \,\middle|\, m = 1, 2, 3, \ldots \right\} \cup [3, \infty).$$

We consider only the case $c = 3m/(m+2)$.

The even part of the superconformal net is identified with the coset net for the inclusion $U(1)_{2m+4} \subset SU(2)_m \otimes U(1)_4$.

The irreducible representations of this local conformal net are labeled with triples (j, k, l) with $0 \le j \le m$, $0 \le k < 2m+4$, $0 \le l < 4$ and $j - k + l \in 2\mathbb{Z}$ with the identification $(j, k, l) = (m-j, k+m+2, l+2)$,

The chiral ring is given by $\{(j, j, 0)\}$ and the spectral flow is by $(0, 1, 1)$.

We classify all $N = 2$ superconformal nets with $c < 3$. More quantum fields give more operators, hence a larger von Neumann algebra. So look for possible extensions of $\{A(I)\}$, where $A(I)$ is generated by (super) stress energy tensor.

We have general theory for such a classification based on α-induction and modular invariants.

In similar classifications of local conformal nets and $N = 1$ superconformal nets, our classification lists consist of simple current extensions, the coset constructions, and the mirror extensions in the sense of Xu.

In the $N = 2$ superconformal case, we have a mixture of the coset construction and the mirror extension.

Gepner model: Make a fifth tensor power of the $N = 2$ superconformal net with $c = 9/5$. Cyclic group actions give an example corresponding to a certain 3-dimensional Calabi-Yau manifold arising from a quintic in $\mathbb{CP}^4$.

This construction gives connection to the **mirror symmetry**. It appears as an isomorphism of two $N = 2$ super Virasoro algebras sending J_n to $-J_n$ and $G_m^{\pm}$ to $G_m^{\mp}$.

Some similarity to the **Moonshine net**: We have some basic net whose representation theory is well understood. Then make its finite tensor power, which should still be a basic net. Then finite group actions produce much more interesting examples.

More noncommutative geometry to be studied:

Longo: Quantum index

Analogy between an elliptic operator and an irreducible representation.

Analogy between the Fredholm index and the Jones index. (Carpi-K-Longo: direct relations between the two for $N = 1$ superconformal nets)

Cohomological aspects

Computations of noncommutative geometric invariants

Entire cyclic cohomology and Jaffe-Lesniewski-Osterwalder cocycle: Possible connections to invariants in superconformal field theory.