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Two-scale Γ -convergence of integral functionals and its application to homogenisation of non-linear high-contrast periodic composites

Abstract An analytical framework is developed for passing to the homogenisation limit in (not necessarily convex) variational problems for composites whose material properties oscillate with a small period ε and that exhibit high contrast of order ε^{-1} between the constitutive, “stress-strain”, response on different parts of the period cell. The approach of this article is based on the concept of “two-scale Γ -convergence”, which is a kind of “hybrid” of the classical Γ -convergence (De Giorgi, E., Franzoni, T.: *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **58**, 842–850, 1975) and the more recent two-scale convergence (Nguetseng, G.: *SIAM J. Math. Anal.* **20**, 608–623, 1989). The present study focuses on a basic high-contrast model, where “soft” inclusions are embedded in a “stiff” matrix. It is shown that the standard Γ -convergence in the L^p -space fails to yield the correct limit problem as $\varepsilon \rightarrow 0$, due to the underlying lack of L^p -compactness for minimising sequences. Using an appropriate two-scale compactness statement as an alternative starting point, the two-scale Γ -limit of the original family of functionals is determined, via a combination of techniques from classical homogenisation, the theory of quasiconvex functions and multiscale analysis. The related result can be thought of as a “non-classical” two-scale extension of the well-known theorem by S. Müller (Müller, S.: *Arch. Rational Mech. Anal.* **99**, 189–212, 1987).

Keywords homogenisation · calculus of variations · high-contrast composites · Γ -convergence · two-scale convergence

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1 Introduction

Recent years have seen a number of advances in the development of various analytical techniques aimed at tackling multi-scale problems where the “classical” homogenisation theorems fail to hold. One prominent example of a problem of this sort is the linear periodic second-order elliptic partial differential equation (PDE) with high-contrast in the coefficients, which was considered by Zhikov [37], see also [6]. Suppose that the period ε and the “contrast” δ^{-1} are scaled so that $\delta \sim \varepsilon^2$ as $\varepsilon \rightarrow 0$. A remarkable feature of the behaviour of the families of equations of this kind is that the two-scale character of the solution persists “in the very limit” as $\varepsilon \rightarrow 0$. Roughly speaking, in the classical two-scale expansion (see *e.g.* [4], [3])

$$u_\varepsilon \sim u_0^\delta(x) + \varepsilon u_1^\delta(x, x/\varepsilon) + \varepsilon^2 u_2^\delta(x, x/\varepsilon) + \dots,$$

where for simplicity we disregard the effects near the boundary¹, all the terms become of the same order as $\varepsilon \rightarrow 0$ (and hence $\delta \rightarrow 0$ as well). This leads to the suggestion that in the expansion (1) u_0 can no longer be assumed independent of the “fast variable” x/ε and must be expected to capture the leading-order oscillations on the scale of the small period of size ε .

The presence of the small-scale resonances in a system of this type can be quantified by proving that the spectrum of the associated operator converges in the sense of Hausdorff to the spectrum of a coupled “two-scale” limiting problem with respect to both the “slow” and “fast” variables. Further, the spectrum of this limiting problem is shown to have an infinite set of “band gaps”, or intervals of frequencies at which wave propagation through the material sample is not allowed. This fact, which at a non-rigorous level had for some years been known to physicists, has caused an explosion of interest in the applied mathematics community towards “non-classical” materials, or “metamaterials”, and continues to be actively explored, both theoretically and experimentally. Among popular applications of the distinctive properties of such “metamaterials” is the design of photonic (electromagnetic) and phononic (elastic) band-gap devices, which are capable of transmitting the corresponding kind of energy at large distances with significantly reduced losses compared to the usual transmission devices. Another recent trend is to exploit such materials for purposes of “invisibility”, or “cloaking”, of nearby objects, see [33], [22], [28].

From the mathematical perspective, the majority of the above developments has been set in the context of linear PDEs, which is traditionally the first class of equations to be explored. In many physics applications (such as the electromagnetics) the linear theory suffices, in some others (such as the high-frequency acoustics) it offers a good approximation to the behaviour of realistic objects. In the present article, however, we aim at the applied contexts that are inherently non-linear, such as that of finite elasticity. It is well known that not only the linearised equations of elasticity but also any “elastic” models emerging from variational integrals with a convex stored-energy function fail to satisfy the principle of material frame indifference and hence have to be discarded for purposes of rigorous analysis.

¹ One way of making rigorous analysis possible in this setting is to consider a problem on a fixed-size torus, or “supercell” as numerical analysts would call it

Most of the methods of classical homogenisation — two-scale asymptotic expansions [4], [3], the Tartar’s “energy method” [31], [19], the method of two-scale convergence ([1], [26]), Γ -convergence ([18], [8], [19]) *etc.*) — have been extended to treat a large class of non-linear problems satisfying a number of technical constitutive assumptions, which have normally included some sort of “non-degeneracy” condition and have thus led to “classical” homogenised limits, *i.e.* those where the fast, “micro-scale”, variable (represented by x/ε in (1) above) is not explicitly present. The lack of a coupling effect between the macroscopic and microscopic length-scales in the limit as $\varepsilon \rightarrow 0$ seems to be inextricably linked to the fact that the kind of compactness argument that underpins the mathematical analysis in each of these contexts involves only functions of the macroscopic variable x . For instance, in the context of the two-scale convergence (see *i.e.* [1], [37]) it can be shown that if an L^p -bounded sequence $u_\varepsilon = u_\varepsilon(x)$ weakly two-scale converges to a function $u(x, y)$ of the slow x and the fast y variables and their gradients ∇u_ε are also L^p -bounded then the limit function u is in fact independent of y and the two-scale limit (after passing to a subsequence if necessary) of the gradients ∇u_ε differs from ∇u by the y -gradient of some two-scale function. Similarly, in the context of the classical Γ -convergence one usually deals with $W^{1,p}$ -bounded sequences (more generally, $W^{k,p}$ -bounded ($k \geq 1$) if one thinks of a higher-order problem) of “almost minimisers”, which are compactly embedded into L^p .

It is possible however to envisage practically motivated non-linear settings where the stored-energy function is no longer non-degenerate, meaning that displacement fields whose gradient is “large” on a significant part of the material are compatible with the requirement that the total energy be finite, and can hence be viable candidates as “almost minimisers”. The context of homogenisation seems to be a perfect environment for exploring such “non-linear metamaterials”: on the one hand, it is a promoted view at the moment that realistic stored-energy functions (and in general, any models) of continuous media are those that are obtained by some sort of averaging (or “homogenisation”, “coarse graining”) procedure from a model at a finer length-scale. On the other hand, the idea of introducing into the equations a high-contrast that is appropriately scaled with the microscopic size, has been a successful one in a number of physically relevant contexts mentioned above. In order to make the two-scale homogenisation work, one then first of all clearly needs a kind of “two-scale compactness” principle such as that employed in linear high-contrast problems. This is typically based on the observation that if an L^p -bounded sequence of functions u_ε weakly two-scale converges to a function $u(x, y)$ (*cf.* the above “classical” compactness example), and the sequence of “non-classical” scaled gradients $\varepsilon \nabla u_\varepsilon$ is L^p -bounded then the function $u(x, y)$ is in fact weakly differentiable in y and, up to selecting a subsequence, the scaled gradients $\varepsilon \nabla u_\varepsilon$ two-scale converge to the y -gradient of u . The subsequent passage to the limit can then be performed on the weak formulation of the problem using *e.g.* the definition of the two-scale convergence, see *e.g.* a recent work [41] treating monotonic nonlinear high-contrast problems (*i.e.* problems whose stored-energy function has monotonic gradient). This procedure of passage to the limit, however, is bound to have an *ad hoc* character in general, since in most genuinely nonlinear contexts different weakly two-scale convergent subsequences will have different limits that *a priori* do not satisfy the same limiting equation. One of the key difficulties in the related analysis is to demonstrate that the weak two-scale

limit of the “fluxes” (*i.e.* the values of the gradient of the stored-energy function on the solution gradient) is the flux of the weak two-scale limit of the original sequences, which in the classical context is a key element of the compensated compactness principle of Murat and Tartar ([25], [31]), see also [40], [38], [11] for recent discussions of the related issues in various PDE contexts.

Following this unoptimistic observation, one is led to review the averaging tools mentioned above and notice that the abstract Γ -convergence approach, being an “ansatz-free” method, does not suffer from the lack of a compensated compactness argument. In its standard form, see [8], it does require the usual L^p -compactness principle but it seems to be extendible, however, to situations where a kind of non-classical “two-scale compactness” holds instead, as outlined above. It is the aim of the present work to provide such an extension. More precisely, consider a bounded domain (*i.e.* a connected open set) $\Omega \subset \mathbb{R}^n$, the “unit cell” $Y := [0, 1]^n$, and a Y -periodic domain $E_1 \subset \mathbb{R}^n$. Let $\varepsilon > 0$, denote by $E_1^\varepsilon = \varepsilon E_1$ the “ ε -contraction” of E_1 , and set $E_0 := \mathbb{R} \setminus E_1$, $E_0^\varepsilon = \varepsilon E_0^\varepsilon$ (see the next section for the precise definition of these sets). We study “high-contrast” variational integrals of the form

$$I^\varepsilon(u) := \int_{\Omega} \left(W_1(\nabla u(x)) \chi_{E_1^\varepsilon}(x) + W_0(\varepsilon \nabla u(x)) \chi_{E_0^\varepsilon}(x) - f(x) \cdot u(x) \right) dx, \quad (1)$$

defined on weakly differentiable *vector* functions u from the space $[L^p(\Omega)]^m$. Here the functions W_1, W_0 represent the constitutive response of the material in question and $f \in [L^q(\Omega)]^m$ is the density of the body forces, $p^{-1} + q^{-1} = 1$, $f \cdot u$ is understood as the usual scalar product of vectors f and u . We shall show that under some technical growth and continuity assumptions on the stored-energy functions $W_i(\varepsilon)$, $i = 0, 1$, the functionals I^ε converge as $\varepsilon \rightarrow 0$ in an appropriately defined sense of “strong two-scale Γ -convergence” to the functional (*cf.* (43))

$$I^{\text{hom}}(u) = \int_{\Omega} \left(W_1^{\text{hom}}(\nabla u(x, y)) + \int_{E_0} QW_0(\nabla_y u(x, y)) dy - f(x) \cdot u(x, y) \right) dx, \quad (2)$$

defined on a certain “two-scale” space of functions $u(x, y)$. Here W_1^{hom} is the classical homogenised stored-energy function (see *e.g.* [19]) for the “perforated” material with the stored-energy function W_1 occupying the domains $\Omega \cap E_1^\varepsilon$ and QW_0 is the quasiconvex envelope (see *e.g.* [17]) of the function W_0 . This is the main result of the paper, which we formulate in Theorems 24 and 26.

Remark 1 The functionals (1) can be considered in conjunction with various boundary conditions on the function u . In what follows, we carry out our analysis for the case of “Neumann”, or “traction-free” equilibrium configurations, where no boundary conditions on $\partial\Omega$ are additionally specified in (1), thus implying the widest possible underlying function space. This kind of setting is often referred to as the “analysis of arbitrary sequences”. It may be pointed out however, that most of other common boundary conditions are also accounted for via a standard procedure (see *e.g.* [8]).

Further details of the problem we investigate are given in Section 2 of the present work. Before proceeding to the two-scale analysis of the functionals (1), in Section 3 we recall the standard definitions of Γ -convergence for sequences of

functionals defined on $L^p(\Omega)$ and two-scale convergence of function sequences in $L^p(\Omega)$. We also provide a version of the “continuous extension lemma in Sobolev spaces on perforated domains, which we use in the subsequent analysis. In Section 4 we demonstrate via a simple example that the usual Γ -convergence does not suffice to obtain a proper variational limit of the functionals (1), and in Section 5 we introduce the notion of two-scale Γ -convergence, which later on is shown to yield the correct (two-scale) limit functional. Sections 6, 7, 8 contain the main technical component of the work, where we prove a statement (Theorem 24) regarding the two-scale Γ -convergence of the sequence (1) as $\varepsilon \rightarrow 0$ to a functional I^{hom} . In Sections 9 and 10 we prove the existence of a minimiser for I^{hom} and the convergence of the infima of the functionals (1) to the minimum of I^{hom} . Finally, Section 11 contains a discussion of the relationship between our results and other topics in applied analysis and of future extensions of our approach.

In conclusion to this introductory section we ought to point out that the technique we present here can be compared against that of the work [9]. The latter considers sequences of high-contrast functionals that are different from (1) in the way the contrast enters the stored-energy function. This difference between the two modelling approaches leads in the case of [9] to somewhat more restricted homogenisation theorems involving limit functionals with respect to the slow variable x only, compared to our results, which provide a “full” two-scale limit (2).

2 Problem setup

We make the following assumptions regarding the functionals (1). The stored-energy functions W_1 and W_0^ε are continuous and satisfy standard growth conditions:

$$\begin{aligned} \alpha|e|^p &\leq W_1(e) \leq \beta(1 + |e|^p), \\ \alpha|e|^p &\leq W_0^\varepsilon(e) \leq \beta(1 + |e|^p), \end{aligned} \quad (3)$$

with $\alpha, \beta > 0$, $p > 1$. [More generally, we will henceforth say that $W(e)$ satisfies the standard growth conditions if $\alpha|e|^p \leq W(e) \leq \beta(1 + |e|^p)$ for some $\alpha, \beta > 0$, where the exponent p is the same as in (3).] Functions W_0^ε are assumed to converge pointwise to some continuous W_0 as $\varepsilon \rightarrow 0$,

$$W_0^\varepsilon(e) \rightarrow W_0(e) \quad (4)$$

for all e as $\varepsilon \rightarrow 0$. In addition W_0^ε must satisfy the following regularity property,

$$|W_0^\varepsilon(e_1) - W_0^\varepsilon(e_2)| \leq \beta(1 + |e_1|^{p-1} + |e_2|^{p-1})|e_1 - e_2| \quad (5)$$

for all e_1, e_2 , which, if W_0^ε is smooth, is equivalent to

$$|\nabla W_0^\varepsilon(e)| \leq C(1 + |e|^{p-1}).$$

Obviously, due to the convergence (4), W_0 possesses the same regularity property:

$$|W_0(e_1) - W_0(e_2)| \leq \beta(1 + |e_1|^{p-1} + |e_2|^{p-1})|e_1 - e_2|. \quad (6)$$

Let Ω be a bounded domain in \mathbb{R}^n with a Lipschitz boundary. Let E_0 be an open Y -periodic set in \mathbb{R}^n such that the closure of $Y_0 := E_0 \cap Y$ is contained in Y , and Y_0

has a Lipschitz boundary, *i.e.* E_0 is a periodic union of open mutually disjoint sets, and let $E_1 = \mathbb{R}^n \setminus \overline{E_0}$. We describe the “soft” component E_0^ε of an elastic composite as the union of all small sets $\varepsilon(Y_0 + z)$, $z \in \mathbb{Z}^n$, such that $\varepsilon(Y + z) \cap (\mathbb{R}^n \setminus \Omega) = \emptyset$. Further, the “stiff” component is defined as $E_1^\varepsilon = \Omega \setminus \overline{E_0^\varepsilon}$. Denote by $\chi_i^\varepsilon(x)$ the characteristic functions of the sets E_i^ε , $i = 1, 2$. We will study the asymptotic properties of the following high-contrast sequence of integral functionals (*cf.* (1)):

$$I^\varepsilon(u) := \int_{\Omega} \left(W_1(\nabla u(x)) \chi_1^\varepsilon(x) + W_0^\varepsilon(\varepsilon \nabla u(x)) \chi_0^\varepsilon(x) \right) dx. \quad (7)$$

Notice that we can drop the “right hand side” $-\int_{\Omega} u(x) \cdot f(x) dx$ part of (1), due to the obvious additivity property of Γ -convergence with respect to the linear terms.

It is clear from the form of the functionals I^ε that as $\varepsilon \rightarrow 0$, on the set E_0^ε much larger values of the gradient ∇u are allowed than on E_1^ε . If we speak about elasticity ($n \times m = 3 \times 3$) this means that the elastic material in E_0^ε is much softer (allows larger deformations) than the material in E_1^ε . Therefore we will use the terms “soft component” and “stiff component” (the latter is often termed “matrix” in the literature on high-contrast problems) when referring to the sets E_0^ε and E_1^ε respectively.

We reserve the notation C for a generic positive constant independent of ε , whose specific value may change from one place to another within the paper or even within a single sequence of relations. Throughout the paper we assume that any function of two variables $f(x, y)$ defined on $\Omega \times Y$ is extended periodically with respect to y into $\Omega \times \mathbb{R}^n$, so that the formal substitution $y = x/\varepsilon$ will make sense for any $x \in \Omega$.

By QW_0 we denote the quasiconvex envelope of W_0 , see *e.g.* [17]. If $V_1 \geq V_2$ then for their quasiconvex envelopes the inequality is preserved, $QV_1 \geq QV_2$. Since the functions $\alpha|e|^p$ and $\beta(1 + |e|^p)$ are quasiconvex and the quasiconvex envelope of a quasiconvex function equals itself, the quasiconvex envelope QW_0 satisfies the standard growth conditions.

Remark 2 There is a reason for taking a sequence W_0^ε rather than a fixed W_0 for describing the soft component of the material. Two natural physical requirements for the stored-energy function is that it must be frame indifferent and attain its minimum value at $e = \nabla u = I$. However, in the case of a soft-component stored-energy function W_0 that is independent of ε these requirements would imply $W_0(\varepsilon R) = 0 = \min W_0$ for any $\varepsilon > 0$ and any rotation R , which is too restrictive, while our setting can easily accommodate this assumptions. For instance, if W_0^ε is given by the formula

$$W_0^\varepsilon(e) = \widehat{W} \left((1 - \varepsilon)I + \sqrt{\varepsilon^T e} \right),$$

which for the scaled gradient takes the form

$$W_0^\varepsilon(\varepsilon \nabla u) = \widehat{W} \left(I + \varepsilon \left(\sqrt{(\nabla u)^T \nabla u} - I \right) \right),$$

where $\widehat{W}(e)$ is a continuous function satisfying the standard growth conditions and attaining its minimum at $e = I$, $\widehat{W}(I) = 0$, then $W_0^\varepsilon(\varepsilon \nabla u)$ is frame indifferent ($W_0^\varepsilon(\varepsilon R \nabla u) = W_0^\varepsilon(\varepsilon \nabla u)$ for any rotation R) and attain its minimum value at

$\nabla u = I$, $W_0^\varepsilon(\varepsilon I) = \widehat{W}(I) = 0$. Note that for any sequence $e_\varepsilon \rightarrow e$ as $\varepsilon \rightarrow 0$ there is convergence

$$W_0^\varepsilon(e_\varepsilon) \rightarrow \widehat{W}\left(I + \sqrt{e^T e}\right) \quad (8)$$

as $\varepsilon \rightarrow 0$, in particular (4) is satisfied with $W_0(e) = \widehat{W}\left(I + \sqrt{e^T e}\right)$.

3 Preliminaries: $\Gamma(L^p)$ -convergence, two-scale convergence and an extension lemma

In what follows (see Section 4) we discuss the standard $\Gamma(L^p)$ -convergence, see [8], applied to the present high-contrast setting. Here we recall its definition and its most important property implying convergence of the infima of Γ -convergent functionals to the minimum of their Γ -limit and convergence of an infimising sequence to a minimiser of the Γ -limit. We also give a definition of two-scale convergence and formulate an extension lemma, which we make use of in the subsequent sections.

Definition 3 Let $F_j : L^p(\Omega) \rightarrow \overline{\mathbb{R}}$ be a sequence of functionals and $F_\infty : L^p(\Omega) \rightarrow \overline{\mathbb{R}}$, where $\Omega \in \mathbb{R}^n$ is bounded and open, and $\overline{\mathbb{R}} := \mathbb{R} \cup +\infty$. We say that F_j $\Gamma(L^p)$ -converge to F_∞ , and write $\Gamma(L^p)$ - $\lim_j F_j = F_\infty$, if for all $u \in L^p(\Omega)$ we have

1. for every sequence $u_j \rightarrow u$ in $L^p(\Omega)$ one has

$$F_\infty(u) \leq \liminf_j F_j(u_j);$$

2. there exists a sequence $u_j \rightarrow u$ in $L^p(\Omega)$ such that

$$F_\infty(u) = \liminf_j F_j(u_j).$$

Remark 4 As is customary in the theory of Γ -convergence, whenever there is a sequence of functionals F_ε parametrised by a ‘‘continuous’’ parameter $\varepsilon \in (0, \varepsilon_0]$, we write $\Gamma(L^p)$ - $\lim_{\varepsilon \rightarrow 0} F_\varepsilon = F_0$ if for any sequence $\varepsilon_j \rightarrow 0$ one has $\Gamma(L^p)$ - $\lim_j F_{\varepsilon_j} = F_0$.

The fundamental requirement for a proper variational convergence is that it must provide the convergence of infima and infimising sequences for F_j to the minimum and a minimiser of the limiting functional F_∞ . The Γ -convergence provides the desired goals given that an additional condition is fulfilled, namely, an appropriate compactness property of infimising sequences. This observation is formulated in the following statement, which is a simple consequence of the definition of $\Gamma(L^p)$ -convergence.

Theorem 5 Let F_j be a sequence of functionals on $L^p(\Omega)$ and suppose that there exists a compact set $K \in L^p(\Omega)$ such that $\inf_{L^p(\Omega)} F_j = \inf_K F_j$ for any j . Let $F_\infty = \Gamma(L^p)$ - $\lim_j F_j$ then

$$\exists \min_{L^p(\Omega)} F_\infty = \lim_j \inf_{L^p(\Omega)} F_j.$$

Moreover, if $u_j \rightarrow u_\infty$ and $\lim_j F_j(u_j) = \lim_j \inf_{L^p(\Omega)} F_j$, then u_∞ is a minimum point for F_∞ .

The requirement of the existence of a compact K on which the infima for all j are attained (mild equicoercivity condition) can be replaced by a weaker one, namely that there exists a compact sequence u_j such that $\liminf_j F_j(u_j) = \liminf_j (\inf_{L^p(\Omega)} F_j)$.

Next, we give the definition of the two-scale convergence of functions. Using the latter will simplify some of the reasoning below. The two-scale convergence will also be essentially employed in the subsequent sections when dealing with two-scale Γ -convergence.

Definition 6 We say that a bounded in $L^p(\Omega)$ sequence v_ε is weakly two-scale convergent to a function $v \in L^p(\Omega \times Y)$, $v_\varepsilon(x) \xrightarrow{2} v(x, y)$, if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v_\varepsilon(x) \varphi(x) b\left(\frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int_Y v(x, y) \varphi(x) b(y) dy dx$$

for all $\varphi \in C_0^\infty(\Omega)$ and all $b \in C_{\text{per}}^\infty(Y)$ (where $C_{\text{per}}^\infty(Y)$ is the set of Y -periodic functions from $C^\infty(\mathbb{R}^n)$).

Definition 7 We say that a bounded in $L^p(\Omega)$ sequence u_ε is strongly two-scale convergent to a function $u \in L^p(\Omega \times Y)$, $u_\varepsilon(x) \xrightarrow{2} u(x, y)$, if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(x) v_\varepsilon(x) dx = \int_{\Omega} \int_Y u(x, y) v(x, y) dy dx$$

for all $v_\varepsilon(x) \xrightarrow{2} v(x, y)$ from $L^q(\Omega)$ and $L^q(\Omega \times Y)$ respectively.

The two-scale convergence, unlike the usual L^p -convergence, allows to keep information about the behaviour of a sequence of functions at the ε -scale when passing to the limit. In this paper we will use some well-known properties of two-scale convergence. The related statements as well as some deeper results can be found *e.g.* in [37].

Finally, throughout the paper we make use of the following extension lemma (*cf.* [8]²)

Lemma 8 *There exists a linear continuous extension operator $T_\varepsilon : W^{1,p}(E_1^\varepsilon) \rightarrow W^{1,p}(\Omega)$ and a constant $C > 0$ such that*

$$\begin{aligned} T_\varepsilon u &= u \text{ in } E_1^\varepsilon, \\ \int_{\Omega} |T_\varepsilon u|^p dx &\leq C \int_{E_1^\varepsilon} |u|^p dx, \\ \int_{\Omega} |\nabla(T_\varepsilon u)|^p dx &\leq C \int_{E_1^\varepsilon} |\nabla u|^p dx, \end{aligned}$$

where the constants C does not depend on ε or Ω .

² Lemma 8 provides somewhat stronger estimates than the extension result of [8], due to the restrictions we impose on the geometry of the set E_1^ε near the boundary of Ω , and is derived directly from Lemma B.6 of [8].

4 $\Gamma(L^p)$ -compactness and its failure in high-contrast homogenisation

In the present section we discuss the compactness of the sequence of high-contrast functionals in the sense of $\Gamma(L^p)$ -convergence, providing an explicit formula for the limit functional in a concrete but quite general case. At the end of the section we will give a simple example illustrating that the lack of compactness for infimising sequences leads to the failure of Γ -convergence to provide a proper variational limit for the related functionals.

It is well known that the Γ -compactness result (*i.e.* existence of a Γ -convergent subsequence) holds for integral functionals of the form

$$F_\varepsilon(u) = \int_{\Omega} f_\varepsilon(x, \nabla u) \, dx$$

with a positive stored-energy function f_ε satisfying the growth estimate from above (but not necessarily from below):

$$0 \leq f_\varepsilon(x, e) \leq \beta(1 + |e|^p),$$

see *e.g.* [8].³ In particular this result holds for high-contrast functionals in the present setting. We formulate it in the following statement.

Theorem 9 *Let I^ε be given by (7). For every positive sequence $\varepsilon_i \rightarrow 0$ there exists a subsequence ε_{i_k} and a measurable function $\varphi \geq 0$ such that*

$$I^0(u) := \int_{\Omega} \varphi(\nabla u) \, dx = \Gamma(L^p)\text{-}\lim_{k \rightarrow \infty} I^{\varepsilon_{i_k}}(u)$$

for all $u \in W^{1,p}(\Omega)$ and all open subsets U of Ω .

Given an additional assumption on W_0 it is even possible to write an explicit formula for I^0 , as follows.

Theorem 10 *Let $W_0(e)$ in (4) be such that it attains its minimum at $e = 0$. Then*

$$I^0(u) = \Gamma(L^p)\text{-}\lim_{\varepsilon \rightarrow 0} I^\varepsilon(u) = \int_{\Omega} W_1^{\text{hom}}(\nabla u) \, dx + |\Omega \times Y_0| W_0(0), \quad (9)$$

where

$$W_1^{\text{hom}}(e) = \liminf_{t \rightarrow \infty} \left\{ \frac{1}{t^n} \int_{(0,t)^n \cap E_1} W_1(e + \nabla \varphi) \, dx : \varphi \in W_0^{1,p}((0,t)^n) \right\}. \quad (10)$$

Proof Notice that the “stiff part” of the Γ -limit (9), (10) is exactly the $\Gamma(L^p)$ -limit for a non-high-contrast integral functional defined on a perforated domain $\Omega \cap E_1^\varepsilon$ (“connected media”, see [8, Part IV]),

$$I_1^\varepsilon(u) := \int_{\Omega} W_1(\nabla u(x)) \chi_1^\varepsilon(x) \, dx.$$

Let us fix some $u \in W^{1,p}(\Omega)$ and consider an arbitrary sequence $u_\varepsilon \in W^{1,p}(\Omega)$ converging to u in $L^p(\Omega)$. There are two possibilities:

³ Although it is not stated explicitly in [8] (on the contrary, the related statement in [8] seems to give an opposite impression), the assumption of the growth estimate from below for the stored-energy function is not used in the proof of the Γ -compactness result.

1. there exists $C > 0$ such that $\|\nabla u_\varepsilon \chi_1^\varepsilon\|_{L^p(\Omega)} + \|\varepsilon \nabla u_\varepsilon \chi_0^\varepsilon\|_{L^p(\Omega)} \leq C$ for small enough ε ;
2. up to a subsequence $\|\nabla u_\varepsilon \chi_1^\varepsilon\|_{L^p(\Omega)} + \|\varepsilon \nabla u_\varepsilon \chi_0^\varepsilon\|_{L^p(\Omega)} \rightarrow \infty$ as $\varepsilon \rightarrow 0$;

Clearly, in the latter case (case 2) we have $I^\varepsilon(u_\varepsilon) \rightarrow \infty$ due to the standard growth conditions. Let us consider the first possibility (case 1). For any ε there exists an extension \tilde{u}_ε of the function $u_\varepsilon \chi_1^\varepsilon$ to the whole Ω such that

$$\|\nabla \tilde{u}_\varepsilon\|_{L^p(\Omega)} \leq C \|\nabla u_\varepsilon \chi_1^\varepsilon\|_{L^p(\Omega)}, \quad (11)$$

where C does not depend on ε , see Lemma 8. Then $\nabla \tilde{u}_\varepsilon$ is bounded in $L^p(\Omega)$. Hence \tilde{u}_ε is weakly compact in $W^{1,p}(\Omega)$ and strongly compact in $L^p(\Omega)$, and there exists $\tilde{u} \in W^{1,p}(\Omega)$ such that $\tilde{u}_\varepsilon \rightarrow \tilde{u}$ in $L^p(\Omega)$ up to a subsequence. By the properties of two-scale convergence we have

$$\begin{aligned} \tilde{u}_\varepsilon \chi_1^\varepsilon &\xrightarrow{2} \tilde{u}(x) \chi_1(y), \\ u_\varepsilon \chi_1^\varepsilon &\xrightarrow{2} u(x) \chi_1(y). \end{aligned}$$

Since $\tilde{u}_\varepsilon \chi_1^\varepsilon = u_\varepsilon \chi_1^\varepsilon$ by definition, we conclude that $\tilde{u}(x) = u(x)$. Due to estimate (11), $\varepsilon \nabla \tilde{u}_\varepsilon \rightarrow 0$ in $L^p(\Omega)$. This implies the convergence of the soft component integral on \tilde{u}_ε :

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} W_0^\varepsilon(\varepsilon \nabla \tilde{u}_\varepsilon(x)) \chi_0^\varepsilon dx = |\Omega \times Y_0| W_0(0). \quad (12)$$

Equality (12) follows from the dominated convergence theorem in view of the standard growth conditions. Thus we have shown that for any sequence $u_\varepsilon \rightarrow u$ in $L^p(\Omega)$ possessing the property 1 there exists a sequence $\tilde{u}_\varepsilon \rightarrow u$ in $L^p(\Omega)$ such that, firstly,

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} I^\varepsilon(\tilde{u}_\varepsilon) &\leq \liminf_{\varepsilon \rightarrow 0} I^\varepsilon(u_\varepsilon), \\ \limsup_{\varepsilon \rightarrow 0} I^\varepsilon(\tilde{u}_\varepsilon) &\leq \limsup_{\varepsilon \rightarrow 0} I^\varepsilon(u_\varepsilon), \end{aligned}$$

and, secondly,

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} I^\varepsilon(\tilde{u}_\varepsilon) &= \liminf_{\varepsilon \rightarrow 0} I_1^\varepsilon(\tilde{u}_\varepsilon) + |\Omega \times Y_0| W_0(0), \\ \limsup_{\varepsilon \rightarrow 0} I^\varepsilon(\tilde{u}_\varepsilon) &= \limsup_{\varepsilon \rightarrow 0} I_1^\varepsilon(\tilde{u}_\varepsilon) + |\Omega \times Y_0| W_0(0). \end{aligned}$$

This, using the definition of Γ -convergence, implies that

$$\Gamma(L^p)\text{-}\lim_{\varepsilon \rightarrow 0} I^\varepsilon = \Gamma(L^p)\text{-}\lim_{\varepsilon \rightarrow 0} I_1^\varepsilon + |\Omega \times Y_0| W_0(0).$$

Combined with the known result (see *e.g.* [8]) on the $\Gamma(L^p)$ -convergence of the sequence I_1^ε this yields

$$\Gamma(L^p)\text{-}\lim_{\varepsilon \rightarrow 0} I^\varepsilon = I^0.$$

□

The Γ -convergence itself without an appropriate compactness property of infimising sequences does not imply the desired convergence of infima of I^ε to the minimum of the $\Gamma(L^p)$ -limit. We illustrate this observation in the following example.

Example 11 Let $W_1(e) = W_0^\varepsilon(e) = |e|^p$ and consider a one-dimensional minimisation problem for the functional

$$I^\varepsilon(u) = \int_{\Omega} (|\partial_x u|^p \chi_1^\varepsilon + \varepsilon^p |\partial_x u|^p \chi_0^\varepsilon) dx - \int_{\Omega} f u dx,$$

where $f \in L^q(\Omega)$ (recall that $p^{-1} + q^{-1} = 1$) is positive and $Y_0 = (1/4, 3/4)$. Then the $\Gamma(L^p)$ -limit for I^ε is given by

$$I^0(u) = |Y_1| \int_{\Omega} |\partial_x u|^p dx - \int_{\Omega} f u dx.$$

Consider also the functional

$$\tilde{I}^0(u) = |Y_1| \int_{\Omega} |\partial_x u_0|^p dx + \int_{\Omega} \int_{Y_0} |\partial_y v|^p dy dx - \int_{\Omega} \int_{Y_0} f v dy dx,$$

acting on functions of the form $u(x, y) = u_0(x) + v(x, y)$, where $u_0 \in W^{1,p}(\Omega)$, $v \in L^p(\Omega, W_0^{1,p}(Y_0))^4$. Next we will show that

$$\inf \tilde{I}^0 < \inf I^0.$$

Since $\tilde{I}^0(u) = I^0(u_0) + G(v)$, where $G(v) = \int_{\Omega} \int_{Y_0} |\partial_y v|^p dy dx - \int_{\Omega} \int_{Y_0} f v dy dx$, we only need to show that $\min G(v) < 0$. Consider the following function $w \in W_0^{1,p}(Y_0)$ extended by zero in Y_1 :

$$w(y) = \frac{1}{4} - \left| x - \frac{1}{2} \right|.$$

Straightforward calculations show that for $\beta > 0$

$$\int_{Y_0} |\alpha(\beta) \partial_y w(y)|^p dy - \beta \int_{Y_0} \alpha(\beta) w(y) dy = -C \beta^q < 0,$$

where

$$\alpha(\beta) = \left(\frac{\beta}{8p} \right)^{q-1},$$

$$C = \frac{1-p}{2(8p)^q} > 0,$$

Then

$$\inf G(v) \leq G(\alpha(f(x))w(y)) = -C \int_{\Omega} (f(x))^p dx < 0,$$

⁴ By $L^p(\Omega, W_0^{1,p}(Y_0))$ we denote the space of functions defined on Ω with values in $W_0^{1,p}(Y_0)$ equipped with the L^p -norm.

since $f(x)$ is positive by the assumption. Hence there exists a function $u = u_0(x) + v(x, y)$ such that

$$\tilde{I}^0(u) < \inf I^0. \quad (13)$$

In particular, this function can be chosen such that it is Carathéodory (that is measurable with respect to x and continuous with respect to y) and $v(x, y)$ is differentiable in x .

Consider now the sequence $u_\varepsilon(x) = u_0(x) + v(x, \varepsilon^{-1}x)$. It follows from the properties of the two-scale convergence, see *e.g.* [37], that

$$\begin{aligned} u_\varepsilon &\xrightarrow{2} u, \\ |\partial_x u_\varepsilon|^p \chi_1^\varepsilon &\xrightarrow{2} |\partial_x u_0|^p \chi_1, \\ \varepsilon^p |\partial_x u_\varepsilon|^p \chi_0^\varepsilon &\xrightarrow{2} |\partial_y v|^p \chi_0. \end{aligned}$$

This implies the convergence

$$I^\varepsilon(u_\varepsilon) \rightarrow \tilde{I}^0(u)$$

as $\varepsilon \rightarrow 0$. Then from (13) it follows that

$$\liminf_{\varepsilon \rightarrow 0} I^\varepsilon < \inf I^0. \quad (14)$$

The above example⁵ indicates that the appropriate type of variational convergence in the high-contrast setting would have to be capable of accounting for rapid oscillations on the soft inclusions “in the very limit” as $\varepsilon \rightarrow 0$. At the same time, we aim at making sure that the requirement of convergence of infima and of minimising sequences is satisfied. As was discussed in Section 3 in the $L^p(\Omega)$ setting, one way to ensure this is to ask for these sequences to be *compact* with respect to the convergence in the underlying function space. It is well-known that the limiting behaviour as $\varepsilon \rightarrow 0$ of sequences u_ε satisfying an *a priori* bound on $\varepsilon \nabla u_\varepsilon$ can be characterised using the notion of two-scale convergence (see Definitions 6 and 7 and [39]). Hence, we extend the usual notion of Γ -convergence to enable one to handle such sequences within the new framework. The result of this extension is the “two-scale Γ -convergence”, which we introduce in the next section. Some ideas in the direction of such an extension have appeared in the literature in other contexts, see [21], [34], [35]. Our main objective is to describe the behaviour of *high-contrast* functionals of the form (1), so we focus on those properties of the two-scale Γ -convergence that are relevant in our setting.

⁵ Strictly speaking, the example does not satisfy the connectedness condition on the stiff component E_1^ε . However, its multi-dimensional analogue, whose construction is clearly similar while somewhat more cumbersome, would comply with all the requirements of Sections 1 and 2 and would still lead to (14), *i.e.* to the failure of the usual $\Gamma(L^p)$ -convergence to pick up the behaviour of minimising sequences for I^ε .

5 Two-scale Γ -convergence

The definition of the two-scale Γ -convergence (for which we use the shorthand Γ_{ts} -convergence) is very similar to the definition of the standard Γ -convergence with the only difference that instead of using the L^p -topology we use the two-scale convergence of functions.

Definition 12 Let $F_\varepsilon : L^p(\Omega) \rightarrow \overline{\mathbb{R}}$ be a sequence of functionals and $F_0 : L^p(\Omega \times Y) \rightarrow \overline{\mathbb{R}}$. We say that F_ε Γ_{ts} -converges to F_0 and write $F_0 = \Gamma_{\text{ts}}\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon$ if for all $u \in L^p(\Omega \times Y)$ we have

1. for every sequence $L^p(\Omega) \ni u_\varepsilon \xrightarrow{2} u$ one has

$$F_0(u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon); \quad (15)$$

2. there exists a sequence $L^p(\Omega) \ni u_\varepsilon \xrightarrow{2} u$ such that

$$F_0(u) = \lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon). \quad (16)$$

One of the important properties of the usual Γ -convergence is that the Γ -limit is lower semicontinuous with respect to the topology of the underlying function (or, more generally, metric) space. The proof of this fact is simple and based on use of the first property in the definition of Γ -convergence and the method of extraction of a diagonal subsequence, see *e.g.* [8, Chapter 7]. In the present scope of the two-scale Γ -convergence the limit functional also possesses the lower semicontinuity property. However, the proof of this fact is not so obvious due to the lack of metric properties for the two-scale convergence. When $u_\varepsilon \rightarrow u$ in $L^p(\Omega)$ one can measure “closeness” of u_ε to u by $\|u_\varepsilon - u\|_{L^p(\Omega)}$. If $u_\varepsilon(x) \xrightarrow{2} u(x, y)$ and u is a Carathéodory function, then one can measure the closeness by $\|u_\varepsilon(x) - u(x, \varepsilon^{-1}x)\|_{L^p(\Omega)}$, but if u is not Carathéodory, *i.e.* $u(x, \varepsilon^{-1}x)$ behaves “badly”, then there is no obvious way of doing this. Next we address the problem of extracting a diagonal sequence in the scope of two-scale convergence and prove the lower semicontinuity of a two-scale Γ -limit.

Whereas the correct passage from the space of two variables to the space of one variable by change $y = x/\varepsilon$ requires some regularity properties of function and not possible in general, the unfolding method (see *e.g.* [14]) allows the passage in the opposite direction while preserving convergence properties. Below we give the definition of the unfolding operator and its weak convergence property from [14]. We denote by $[y]$ the integer part of the vector y (that is each component of $[y]_i$ is the largest integer such that $[y]_i \leq y_i$, thus $y - [y] \in Y$).

Definition 13 For Lebesgue-measurable functions φ on Ω , the unfolding operator \mathcal{T}_ε is defined as follows:

$$\mathcal{T}_\varepsilon(\varphi)(x, y) = \varphi\left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon y\right) \text{ for } (x, y) \in \Omega \times Y.$$

Proposition 14 Let w_ε be a bounded sequence in $L^p(\Omega)$. The following statements are equivalent:

1. $w_\varepsilon \xrightarrow{2} w \in L^p(\Omega \times Y)$,
2. $\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w$ in $L^p(\Omega \times Y)$.

The above proposition immediately implies the following equivalence of convergence properties.

Proposition 15 *A sequence w_ε from $L^p(\Omega)$ strongly two-scale converges to $w \in L^p(\Omega \times Y)$, $w_\varepsilon \xrightarrow{2} w$, if and only if $\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w$ in $L^p(\Omega \times Y)$.*

Proof Let $w_\varepsilon \xrightarrow{2} w$. Then $w_\varepsilon \xrightarrow{2} w$, $\|w_\varepsilon\|_{L^p(\Omega)} \rightarrow \|w\|_{L^p(\Omega \times Y)}$ by the properties of the two-scale convergence, in particular w_ε is bounded in $L^p(\Omega)$. By Proposition 14

$$\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w \text{ in } L^p(\Omega \times Y). \quad (17)$$

Also, by the properties of unfolding operator $\|w_\varepsilon\|_{L^p(\Omega)} = \|\mathcal{T}_\varepsilon(w_\varepsilon)\|_{L^p(\Omega \times Y)}$, hence we have the convergence of norms $\|\mathcal{T}_\varepsilon(w_\varepsilon)\|_{L^p(\Omega \times Y)} \rightarrow \|w\|_{L^p(\Omega \times Y)}$. Together with weak convergence (17) the latter implies $\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w$ in $L^p(\Omega \times Y)$. The proof of the opposite statement is analogous. \square

Remark 16 The above proposition ensures the metric properties of the strong two-scale convergence. Namely, $w_\varepsilon \xrightarrow{2} w$ if and only if $\lim_{\varepsilon \rightarrow 0} \|\mathcal{T}_\varepsilon(w_\varepsilon) - w\|_{L^p(\Omega \times Y)} = 0$. Thus $\|\mathcal{T}_\varepsilon(w_\varepsilon) - w\|_{L^p(\Omega \times Y)}$ indeed measures a ‘‘distance’’ between w_ε and w at least to the extent to which a distance between elements of different spaces can be measured.

Proposition 17 *Let $u^k(x, y) \rightarrow u(x, y)$ in $L^2(\Omega \times Y)$. Suppose that for any $k \in \mathbb{N}$ there exists a sequence $L^p(\Omega) \ni v_\varepsilon^k(x) \xrightarrow{2} u^k(x, y)$ as $\varepsilon \rightarrow 0$. Let ε'_k be some given positive sequence. Then there exists a sequence $\varepsilon_k \rightarrow 0$, with the property $\varepsilon_k \leq \varepsilon'_k$, and such that $v_{\varepsilon_k}^k(x) \xrightarrow{2} u(x, y)$.*

Proof Proposition 15 makes the proof elementary. Let us apply the unfolding operator to the sequences v_ε^k . We have $\mathcal{T}_\varepsilon(v_\varepsilon^k) \rightarrow u^k(x, y)$ in $L^2(\Omega \times Y)$ for any k . One can extract a diagonal subsequence $\mathcal{T}_{\varepsilon_k}(v_{\varepsilon_k}^k)$ (with $\varepsilon_k \rightarrow 0$) converging to u in $L^2(\Omega \times Y)$. Then by Proposition 15 we obtain $v_{\varepsilon_k}^k(x) \xrightarrow{2} u(x, y)$. \square

Theorem 18 *Let $F_0 = \Gamma_{\text{is}}\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon$. Then F_0 is lower semicontinuous with respect to the strong convergence in $L^p(\Omega \times Y)$.*

Proof Let $u^k \rightarrow u$ in $L^p(\Omega \times Y)$. By (16) for each $k \in \mathbb{N}$ there exists a sequence $v_\varepsilon^k(x) \xrightarrow{2} u^k(x, y)$ such that $\lim_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon^k) = F_0(u^k)$. Let a sequence ε'_k be chosen such that for every k

$$|F_\varepsilon(v_\varepsilon^k) - F_0(u^k)| \leq 1/k \quad (18)$$

as soon as $\varepsilon \leq \varepsilon'_k$. By Proposition 17 there is a sequence ε_k , $\varepsilon_k \leq \varepsilon'_k$, with the property that $v_{\varepsilon_k}^k \xrightarrow{2} u$. Then by (15) and (18) we have

$$F_0(u) \leq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(v_{\varepsilon_k}^k) = \liminf_{k \rightarrow \infty} F_0(u^k).$$

\square

The definition of Γ_{ts} -convergence is quite general, it gives no information even on the structure of the function space on which the limit functional is defined (*i.e.* attains finite values). In order to understand this we need to consider a sequence of functions with bounded energy and pass to the limit. Let u_ε be such that $I^\varepsilon(u_\varepsilon) \leq C$, for small enough ε for some $C > 0$, where the sequence u_ε is bounded in $L^p(\Omega)$ and the functionals I^ε are introduced in Section 2. The standard growth conditions (3) imply that

$$\|\nabla u_\varepsilon \chi_1^\varepsilon\|_{L^p(\Omega)} + \|\varepsilon \nabla u_\varepsilon \chi_0^\varepsilon\|_{L^p(\Omega)} \leq C. \quad (19)$$

By the compactness property of two-scale convergence (see *e.g.* [39]⁶) we have

$$\begin{aligned} u_\varepsilon &\xrightarrow{2} u(x, y) \in L^p(\Omega, W_{\text{per}}^{1,p}(Y)), \\ \varepsilon \nabla u_\varepsilon &\xrightarrow{2} \nabla_y u(x, y), \end{aligned}$$

where $W_{\text{per}}^{1,p}(Y)$ is the Sobolev space of functions periodic on Y . Clearly the characteristic function $\chi_1^\varepsilon(x)$ strongly two-scale converges to $\chi_1(y)$. Hence, by the properties of two-scale convergence

$$\begin{aligned} \chi_1^\varepsilon u_\varepsilon &\xrightarrow{2} \chi_1(y) u(x, y), \\ \varepsilon \chi_1^\varepsilon \nabla u_\varepsilon &\xrightarrow{2} \chi_1(y) \nabla_y u(x, y). \end{aligned} \quad (20)$$

Then from $\|\varepsilon \chi_1^\varepsilon \nabla u_\varepsilon\|_{L^p(\Omega)} \rightarrow 0$ (*cf.* (19)) it follows that $\chi_1(y) \nabla_y u(x, y) \equiv 0$, that is $u(x, y)$ does not depend on y when $y \in Y_1$,

$$\chi_1(y) u(x, y) = \chi_1(y) u_0(x). \quad (21)$$

This observation allows to represent the limit function in the form

$$u(x, y) = u_0(x) + v(x, y),$$

where $v(x, y) \in L^p(\Omega, W_0^{1,p}(Y_0))$.

Next we show that $u_0(x) \in W^{1,p}(\Omega)$. We denote an extension of u_ε by Lemma 8 by \tilde{u}_ε , let also $v_\varepsilon = u_\varepsilon - \tilde{u}_\varepsilon$. Lemma 8 and (19) imply that the sequence \tilde{u}_ε is bounded in $W^{1,p}(\Omega)$, and, hence, converges to some $\tilde{u}_0 \in W^{1,p}(\Omega)$ strongly in $L^p(\Omega)$ (at least up to a subsequence). By the properties of two-scale convergence we have

$$\chi_1^\varepsilon u_\varepsilon = \chi_1^\varepsilon \tilde{u}_\varepsilon \xrightarrow{2} \chi_1(y) \tilde{u}_0(x).$$

Comparing the latter to (20), (21) we conclude that $u_0 = \tilde{u}_0 \in W^{1,p}(\Omega)$.

We next split the functional I^ε in two terms corresponding to the stiff and soft components,

$$I^\varepsilon(u) = I_1^\varepsilon + I_0^\varepsilon := \int_\Omega W_1(\nabla u) \chi_1^\varepsilon \, dx + \int_\Omega W_0^\varepsilon(\varepsilon \nabla u) \chi_0^\varepsilon \, dx, \quad (22)$$

and consider each term separately when passing to the limit as $\varepsilon \rightarrow 0$. Finding the Γ_{ts} -limit of the ‘‘stiff’’ component I_1^ε is straightforward due to the fact that its Γ_{ts} -limit coincides with the $\Gamma(L^p)$ -limit of the perforated media functional studied in

⁶ The corresponding statement in [39] is proven for the case L^2 . However, it remains valid for the general L^p , the related proof being identical.

[8]. On the contrary, derivation of the Γ_{ts} -limit of the “soft” component I_0^ε is non-trivial, it involves using properties of two-scale convergence, periodic unfolding method, properties of quasiconvexity and multi-scale analysis. The basic idea is that the argument $\varepsilon \nabla u_\varepsilon$ of I_0^ε can be written as $\varepsilon \nabla \tilde{u}_\varepsilon + \varepsilon \nabla v_\varepsilon$, where the first term converges to zero in the L^p -norm (*cf.* the discussion above), it allows us to consider I_0^ε on the family of sequences $v_\varepsilon \in W_0^{1,p}(E_0^\varepsilon)$ strongly two-scale converging to elements of $L^p(\Omega, W_0^{1,p}(Y_0))$. Then we will show that passing to the limit on the stiff and soft components separately and then summing up the limits yields the correct Γ_{ts} -limit for the full problem. In the next section we implement the first step in the described strategy.

6 Two-scale Γ -limit of the stiff component

Our focus in the present work is on extending the results of [37], [9], [41], in terms of deriving two-scale limit problems (as $\varepsilon \rightarrow 0$) for a general class of high-contrast integral functionals. An essential element of the related settings is the presence of a main, “matrix”, material, whose properties retain the classical, non-contrast behaviour in all directions at each point of the ambient space. This implies, in particular, that, once a certain extension procedure is carried out, the passing to the limit can be treated separately on the “matrix” (“stiff”) and on the “inclusions” (“soft”). In the present section we determine the “stiff” part of the two-scale limit, which does not present any substantial technical challenges, apart from the need to justify the procedure of splitting the functional into the “stiff” and “soft” components (see Section 8).

We consider the stiff part of the functional, $I_1^\varepsilon = \int_\Omega W_1(\nabla u) \chi_1^\varepsilon \, dx$, defined on functions from $W^{1,p}(E_1^\varepsilon)$. Analogously to the above a bounded energy sequence $u_\varepsilon \in W^{1,p}(E_1^\varepsilon)$ can be shown to converge to some $u \in W^{1,p}(\Omega)$ strongly in $L^p(\Omega)$ up to a subsequence. At the same time, $u_\varepsilon \chi_1^\varepsilon \xrightarrow{2} u(x) \chi_1(y)$. Thus the Γ_{ts} -convergence of I_1^ε is equivalent to its $\Gamma(L^p)$ -convergence in the sense of [8], the following theorem ([24], [7]) is valid.

Theorem 19 *Let the stored-energy function W_1 satisfy the conditions of Section 2. Then the two-scale Γ -limit of the functional I_1^ε is given by*

$$I_1^{\text{hom}}(u) := \Gamma_{\text{ts}}\text{-}\lim_{\varepsilon \rightarrow 0} I_1^\varepsilon(u) = \int_\Omega W_1^{\text{hom}}(\nabla u) \, dx, \quad (23)$$

where

$$W_1^{\text{hom}}(e) = \liminf_{t \rightarrow \infty} \left\{ \frac{1}{t^n} \int_{(0,t)^n \cap E_1} W_1(e + \nabla \varphi) \, dx : \varphi \in W_0^{1,p}((0,t)^n) \right\},$$

and $u \in W^{1,p}(\Omega)$.

7 Two-scale Γ -limit of the soft component

7.1 Lower semi-continuity type result

In this section we will consider the “soft inclusions functional” (cf. (22))

$$I_0^\varepsilon(u) = \int_{\Omega} W_0^\varepsilon(\varepsilon \nabla u) \chi_0^\varepsilon \, dx \quad (24)$$

defined on functions from $W_0^{1,p}(E_0^\varepsilon)$. We will need the following auxiliary lemma, which is similar to the one in [17] where it was employed for the relaxation of non-convex integral functionals. (We use the notation “ \rightharpoonup ” and “ \rightarrow ” for weak and strong convergence in corresponding functional spaces.)

Lemma 20 *Let $D, E \in \mathbb{R}^n$ be cubes with edges parallel to the axes, $e \in \mathbb{R}^{n \times m}$ and $f(e)$ be quasiconvex in e and satisfy the standard growth condition. Let a sequence $z_\varepsilon = z_\varepsilon(x, y) \in L^p(D, W^{1,p}(E))$ be such that $\nabla_y z_\varepsilon(x, y) \rightharpoonup 0$ and $z_\varepsilon \rightarrow 0$ in $L^p(D \times E)$. Then*

$$\liminf_{\varepsilon \rightarrow 0} \int_{D \times E} f(e + \nabla_y z_\varepsilon) \, dy \, dx \geq \int_{D \times E} f(e) \, dy \, dx \quad (25)$$

Proof Let K be an integer and let $E_0 \subset E_1 \subset \dots \subset E_K \subset E$, be cubes such that the centres of E_k and E coincide for any $k = 0, 1, \dots, K$, and

$$\begin{aligned} \frac{1}{2} \text{dist}(E_0, \partial E) &= R, \\ \text{dist}(E_0, \partial E_k) &= \frac{k}{K} R, \end{aligned}$$

for some $R > 0$. Let $\varphi_k(y) \in C^\infty(E)$ be scalar cut-off functions:

$$0 \leq \varphi_k \leq 1, \quad |\nabla \varphi_k| \leq \frac{\tilde{C}K}{R}, \quad \varphi_k(y) = \begin{cases} 1 & \text{if } y \in E_{k-1}, \\ 0 & \text{if } y \in E - E_k, \end{cases}$$

with some $\tilde{C} > 0$. Let

$$z_\varepsilon^k = z_\varepsilon \varphi_k,$$

hence $z_\varepsilon^k \in L^p(D, W_0^{1,p}(E))$. Then we can use quasiconvexity of f to obtain

$$\begin{aligned} & \int_{D \times E} f(e) \, dy \, dx \leq \int_{D \times E} f(e + \nabla_y z_\varepsilon^k) \, dy \, dx \\ & \leq \int_{D \times (E - E_k)} f(e) \, dy \, dx + \int_{D \times (E_k - E_{k-1})} f(e + \nabla_y z_\varepsilon^k) \, dy \, dx \\ & \quad + \int_{D \times E_{k-1}} f(e + \nabla_y z_\varepsilon) \, dy \, dx. \end{aligned}$$

The last estimate immediately implies

$$\int_{D \times E_k} f(e) \, dy \, dx \leq \int_{D \times E} f(e + \nabla_y z_\varepsilon) \, dy \, dx + \int_{D \times (E_k - E_{k-1})} f(e + \nabla_y z_\varepsilon^k) \, dy \, dx. \quad (26)$$

We estimate the last term in (26) using the growth assumptions on f and the properties of φ_k , as follows:

$$\begin{aligned} \int_{D \times (E_k - E_{k-1})} f(e + \nabla_y z_\varepsilon^k) \, dy \, dx &\leq \beta \int_{D \times (E_k - E_{k-1})} (1 + |e + \nabla_y z_\varepsilon^k|^p) \, dy \, dx \\ &\leq C\beta \int_{D \times (E_k - E_{k-1})} (1 + |e|^p + |\varphi_k \nabla_y z_\varepsilon + \nabla_y \varphi_k \otimes z_\varepsilon|^p) \, dy \, dx \\ &\leq C\beta \int_{D \times (E_k - E_{k-1})} (1 + |e|^p + |\nabla_y z_\varepsilon|^p + (\tilde{C}K/R)^p |z_\varepsilon|^p) \, dy \, dx. \end{aligned}$$

Let us take a sum of inequalities (26) for k from 1 to K ,

$$\begin{aligned} f(e) \sum_{k=1}^K |D \times E_k| &\leq K \int_{D \times E} f(e + \nabla_y z_\varepsilon) \, dy \, dx \\ + C \int_{D \times (E_K - E_0)} &(1 + |e|^p + |\nabla_y z_\varepsilon|^p + (\tilde{C}K/R)^p |z_\varepsilon|^p) \, dy \, dx. \end{aligned}$$

Using the fact that $z_\varepsilon \rightarrow 0$ in $L^p(D \times E)$ yields

$$\begin{aligned} f(e) \sum_{k=1}^K |D \times E_k| &\leq K \liminf_{\varepsilon \rightarrow 0} \int_{D \times E} f(e + \nabla_y z_\varepsilon) \, dy \, dx \\ + \limsup_{\varepsilon \rightarrow 0} C \int_{D \times (E_K - E_0)} &(1 + |e|^p + |\nabla_y z_\varepsilon|^p) \, dy \, dx, \end{aligned} \quad (27)$$

where the second term on the right hand side is bounded uniformly in K, R . Hence dividing (27) by K and letting $K \rightarrow \infty$ and $R \rightarrow 0$ we obtain (25). \square

In what follows we will prove the lower semicontinuity type inequality (15) for the soft inclusions functional I_0^ε .

Lemma 21 *Let v_ε be such that $\varepsilon \nabla v_\varepsilon \xrightarrow{2} \nabla_y v(x, y)$ and $v_\varepsilon \xrightarrow{2} v(x, y)$, where $v_\varepsilon \in W_0^{1,p}(E_0^\varepsilon)$ and $v \in L^p(\Omega, W_0^{1,p}(Y_0))$. Then*

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} W_0^\varepsilon(\varepsilon \nabla v_\varepsilon) \chi_0^\varepsilon \, dx \geq \int_{\Omega} \int_{Y_0} QW_0(\nabla_y v) \, dy \, dx,$$

where QW_0 denotes the quasiconvex envelope of W_0 , i.e. the supremum of all quasiconvex functions not exceeding W_0 .

Proof In this proof we mainly follow [17], essentially employing, however, multi-scale analysis. First we use the unfolding operator \mathcal{T}_ε , see Definition 13, in order to pass to the space of two variables. As it was pointed out in Section 5, an important property of \mathcal{T}_ε is that it preserves the convergence properties of function sequences. Namely, let w_ε be bounded in $L^p(\Omega)$, then $w_\varepsilon(x) \xrightarrow{2} w(x, y)$ if and only if $\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w(x, y)$ in $L^p(\Omega \times Y)$ and analogously for the strong convergence. \mathcal{T}_ε transforms derivatives with respect to x into the derivatives with respect to y : $\mathcal{T}_\varepsilon(\varepsilon \nabla w(x)) = \nabla_y(\mathcal{T}_\varepsilon(w)(x, y))$.

Let us denote $\tilde{v}_\varepsilon(x, y) = \mathcal{T}_\varepsilon(v_\varepsilon)$. Then $\tilde{v}_\varepsilon \rightharpoonup v$, $\nabla_y \tilde{v}_\varepsilon \rightharpoonup \nabla_y v$. It is easy to see by a direct calculation that

$$\int_{\Omega} W_0^\varepsilon(\varepsilon \nabla v_\varepsilon) \chi_0^\varepsilon \, dx = \int_{\Omega} \int_{Y_0} W_0^\varepsilon(\nabla_y \tilde{v}_\varepsilon) \, dy \, dx.$$

Since $W_0^\varepsilon(e) \geq QW_0^\varepsilon(e)$, $\forall e$, and hence,

$$\int_{\Omega} W_0^\varepsilon(\varepsilon \nabla v_\varepsilon) \chi_0^\varepsilon \, dx \geq \int_{\Omega} \int_{Y_0} QW_0^\varepsilon(\nabla_y \tilde{v}_\varepsilon) \, dy \, dx,$$

it suffices to show the lower semicontinuity property for the two-scale integral of a quasiconvex function:

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} \int_{Y_0} QW_0^\varepsilon(\nabla_y \tilde{v}_\varepsilon) \, dy \, dx \geq \int_{\Omega} \int_{Y_0} QW_0(\nabla_y v) \, dy \, dx.$$

To this end we cover Ω by a (finite) union of disjoint cubes D_s whose edge length is h , $h > 0$, so that $\Omega \subseteq \bigcup_s D_s$ and $\Omega \cap D_s \neq \emptyset$, $\forall s$. We cover Y_0 by a union of disjoint cubes E_t with the analogous properties. Let us denote by K_{st} the part of $D_s \times E_t$ lying in $\Omega \times Y_0$, $K_{st} := (D_s \times E_t) \cap (\Omega \times Y_0)$. We then take the average of $\nabla_y v$ over each K_{st} ,

$$e_{s,t} = \frac{1}{|K_{st}|} \int_{K_{st}} \nabla_y v \, dy \, dx. \quad (28)$$

Notice first that choosing h small enough we can ensure that

$$\sum_{s,t} \int_{K_{st}} |\nabla_y v - e_{s,t}|^p \, dy \, dx \leq \delta \quad (29)$$

for an arbitrarily small $\delta > 0$. Let us denote

$$\begin{aligned} J^\varepsilon(w) &= \int_{\Omega} \int_{Y_0} QW_0^\varepsilon(\nabla_y w) \, dy \, dx, \\ J(w) &= \int_{\Omega} \int_{Y_0} QW_0(\nabla_y w) \, dy \, dx, \end{aligned}$$

and consider

$$\begin{aligned}
& J^\varepsilon(\nabla_y \tilde{v}_\varepsilon) - J(\nabla_y v) \\
&= \sum_{s,t} \int_{K_{st}} (QW_0^\varepsilon(\nabla_y v + (\nabla_y \tilde{v}_\varepsilon - \nabla_y v)) - QW_0^\varepsilon(e_{s,t} + (\nabla_y \tilde{v}_\varepsilon - \nabla_y v))) \, dy \, dx \\
&\quad + \sum_{s,t} \int_{K_{st}} (QW_0^\varepsilon(e_{s,t} + (\nabla_y \tilde{v}_\varepsilon - \nabla_y v)) - QW_0^\varepsilon(e_{s,t})) \, dy \, dx \\
&+ \sum_{s,t} \int_{K_{st}} (QW_0^\varepsilon(e_{s,t}) - QW_0(e_{s,t})) \, dy \, dx + \sum_{s,t} \int_{K_{st}} (QW_0(e_{s,t}) - QW_0(\nabla_y v)) \, dy \, dx \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

The quasiconvex envelope QW_0^ε is rank-one convex and satisfies the same growth conditions as W_0^ε . Hence it satisfies the following inequality (see [17, Prop. 2.32]),

$$|QW_0^\varepsilon(e_1) - QW_0^\varepsilon(e_2)| \leq \beta(1 + |e_1|^{p-1} + |e_2|^{p-1})|e_1 - e_2|$$

for any $e_1, e_2 \in \mathbb{R}^{n \times m}$, and analogously for QW_0 . Using this fact we can estimate the terms J_1 and J_4 , as follows. Using the Hölder inequality, the L^p boundedness of $\nabla_y \tilde{v}_\varepsilon$ and (29) yields

$$\begin{aligned}
|J_1| &\leq \sum_{s,t} \int_{K_{st}} \beta(1 + |\nabla_y \tilde{v}_\varepsilon|^{p-1} \\
&\quad + |e_{s,t} + (\nabla_y \tilde{v}_\varepsilon - \nabla_y v)|^{p-1}) |\nabla_y v - e_{s,t}| \, dy \, dx \leq \delta
\end{aligned}$$

for small enough h , where we keep the same notation δ as in (29) for an arbitrary small constant. Analogously,

$$|J_4| \leq \delta$$

for small enough h . Now applying Lemma 20 we conclude that

$$\liminf_{\varepsilon \rightarrow 0} J_2 \geq 0.$$

The term J_3 vanishes as $\varepsilon \rightarrow 0$ due to the pointwise convergence of QW_0^ε to QW_0 , which clearly follows from the convergence of W_0^ε to W_0 and the definition of the quasiconvex envelope. Hence we obtain that

$$\liminf_{\varepsilon \rightarrow 0} J^\varepsilon(\nabla_y \tilde{v}_\varepsilon) - J(\nabla_y v) \geq -2\delta,$$

where $\delta > 0$ is arbitrary small. The statement of the lemma then follows immediately. \square

7.2 Existence of a recovery sequence

Lemma 22 *Let $v(x, y) \in L^p(\Omega, W_0^{1,p}(Y_0))$. Then there exists a sequence $v_\varepsilon(x) \in W_0^{1,p}(E_0^\varepsilon)$ such that $v_\varepsilon \xrightarrow{2} v$ and*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} W_0^\varepsilon(\varepsilon \nabla v_\varepsilon) \chi_0^\varepsilon dx = \int_{\Omega} \int_{Y_0} QW_0(\nabla_y v) dy dx. \quad (30)$$

Proof Assume first that $v \in C^\infty(\Omega \times Y_0)$. As in Lemma 21 for each ε we cover the sets Ω and Y_0 by finite unions of disjoint cubes D_s and E_t respectively, with edge length equal to ε . For each ε we introduce the function

$$\tilde{v}_\varepsilon(x, y) = \begin{cases} \frac{1}{|\varepsilon Y|} \int_{D_s \ni x} v(\zeta, y) d\zeta, & x \in D_s \subset \overline{\Omega}, \\ 0 & x \in D_s : D_s \cap \mathbb{R}^n \setminus \overline{\Omega} \neq \emptyset, \end{cases}$$

i.e. for each x we take the mean of v over the cube D_s that contains x . The restriction of $\tilde{v}_\varepsilon(x, y)$ to each D_s is a function of y only. Clearly

$$\begin{aligned} \|\tilde{v}_\varepsilon - v\|_{L^p(\Omega \times Y_0)} &\rightarrow 0, \\ \|\nabla_y \tilde{v}_\varepsilon - \nabla_y v\|_{L^p(\Omega \times Y_0)} &\rightarrow 0. \end{aligned} \quad (31)$$

Notice that the function $\tilde{v}_\varepsilon(x, x/\varepsilon)$ is a well-defined measurable function which belongs to⁷ $W_0^{1,p}(E_0^\varepsilon)$. In the context of the unfolding method (see Definition 13) one has $\mathcal{T}_\varepsilon(\tilde{v}_\varepsilon(x, x/\varepsilon)) = \tilde{v}_\varepsilon(x, y)$. Thus we have

$$\begin{aligned} \int_{\Omega \times Y_0} |\tilde{v}_\varepsilon(x, y)|^p dy dx &= \int_{E_0^\varepsilon} |\tilde{v}_\varepsilon(x, x/\varepsilon)|^p dx, \\ \int_{\Omega \times Y_0} \tilde{v}_\varepsilon(x, y) g(x, y) dy dx &= \int_{E_0^\varepsilon} \tilde{v}_\varepsilon(x, x/\varepsilon) g(x, x/\varepsilon) dx, \end{aligned} \quad (32)$$

for any smooth function g . This and (31) implies that

$$\tilde{v}_\varepsilon(x, x/\varepsilon) \xrightarrow{2} v(x, y)$$

as $\varepsilon \rightarrow 0$.

Further, we approximate the function v by a piecewise constant function $e_{s,t}^\varepsilon = e_{s,t}$ given by (28) with $h = \varepsilon$.⁸ It is well known, see *e.g.* [17, Theorem 6.9], that for any $\delta > 0$ (in particular for $\delta = \varepsilon$) and any $e_{s,t}^\varepsilon$ there exists $\varphi_{s,t}^\varepsilon \in W_0^{1,\infty}(Y)$ such that

$$\left| QW_0^\varepsilon(e_{s,t}^\varepsilon) - \int_Y W_0^\varepsilon(e_{s,t}^\varepsilon + \nabla \varphi_{s,t}^\varepsilon(z)) dz \right| \leq \delta = \varepsilon. \quad (33)$$

⁷ In the above construction, the cover of Ω needs to be taken to consist of the cubes $D_s := \varepsilon(Y + z_s)$, where z_s is an integer vector.

⁸ Obviously, $e_{s,t}$ in (28) depends on h , but there we dropped it in the notation. Now, however, we would like to emphasise the dependence of the approximation $e_{s,t}^\varepsilon$ from ε .

Notice that due to the smoothness of v the means $e_{s,t}^\varepsilon$ are bounded uniformly in ε, s and t . Hence the values $QW_0^\varepsilon(e_{s,t}^\varepsilon)$ are uniformly bounded as well. Therefore the functions $\varphi_{s,t}^\varepsilon$ are uniformly bounded in $W^{1,p}(Y)$, which follows from (33), the growth conditions on W_0^ε and the Poincaré inequality. A change of variables $z = y/\varepsilon$ yields

$$\int_Y W_0^\varepsilon(e_{s,t}^\varepsilon + \nabla \varphi_{s,t}^\varepsilon(z)) dz = \varepsilon^{-n} \int_{\varepsilon Y} W_0^\varepsilon(e_{s,t}^\varepsilon + \varepsilon \nabla_y \varphi_{s,t}^\varepsilon(y/\varepsilon)) dy. \quad (34)$$

Let us denote by $\psi^\varepsilon(x, y) \in L^p(\Omega, W_0^{1,p}(Y_0))$ the function whose restriction to the set $D_s \times E_t$ equals $\varepsilon \varphi_{s,t}^\varepsilon(y/\varepsilon + z_t^\varepsilon)$ if $E_t \times D_s \subset \bar{Y}_0 \times \bar{\Omega}$ and zero otherwise, where the constant vector z_t^ε is chosen so that $\varepsilon Y + z_t^\varepsilon = E_t$ in order to have $\varepsilon \varphi_{s,t}^\varepsilon(y/\varepsilon + z_t^\varepsilon) \in W_0^{1,p}(E_t)$. Obviously $\psi^\varepsilon \rightarrow 0$ in $L^p(\Omega \times Y_0)$. The fact that $\varphi_{s,t}^\varepsilon \in W_0^{1,\infty}(Y)$ and the boundedness of $\nabla_y \psi^\varepsilon$ in $L^p(\Omega \times Y_0)$ imply $\nabla_y \psi^\varepsilon \rightarrow 0$ in $L^p(\Omega \times Y_0)$. From (33) and (34) it follows that

$$\left| \sum_{s,t} \int_{K_{st}} (QW_0^\varepsilon(e_{s,t}^\varepsilon) - W_0^\varepsilon(e_{s,t}^\varepsilon + \nabla_y \psi^\varepsilon)) dy dx \right| \leq C\varepsilon, \quad (35)$$

where the constant C depends on the function v and the sets Ω and Y_0 . Clearly the function $\psi^\varepsilon(x, x/\varepsilon) \in W_0^{1,p}(E_0^\varepsilon)$ is measurable and satisfies equalities analogous to (32). Hence $\psi^\varepsilon(x, x/\varepsilon) \xrightarrow{2} 0$.

Consider the function $v_\varepsilon(x) = \tilde{v}_\varepsilon(x, x/\varepsilon) + \psi^\varepsilon(x, x/\varepsilon)$. We claim that it satisfies the lemma. Indeed, the conditions $v_\varepsilon(x) \in W_0^{1,p}(E_0^\varepsilon)$ and $v_\varepsilon \xrightarrow{2} v$ are obviously fulfilled. In order to prove (30) we notice first that

$$\begin{aligned} & \int_\Omega W_0^\varepsilon(\varepsilon \nabla v_\varepsilon) \chi_0^\varepsilon dx - \int_\Omega \int_{Y_0} QW_0(\nabla_y v) dy dx \\ &= \int_\Omega W_0^\varepsilon(\varepsilon \nabla v_\varepsilon) \chi_0^\varepsilon dx - \sum_{s,t} \int_{K_{st}} W_0^\varepsilon(e_{s,t}^\varepsilon + \nabla_y \psi^\varepsilon) dy dx \\ & \quad + \sum_{s,t} \int_{K_{st}} (W_0^\varepsilon(e_{s,t}^\varepsilon + \nabla_y \psi^\varepsilon) - QW_0^\varepsilon(e_{s,t}^\varepsilon)) dy dx \\ & \quad + \sum_{s,t} \int_{K_{st}} (QW_0^\varepsilon(e_{s,t}^\varepsilon) - QW_0^\varepsilon(\nabla_y v)) dy dx \\ & \quad + \int_{\Omega \times Y_0} (QW_0^\varepsilon(\nabla_y v) - QW_0(\nabla_y v)) dy dx \\ & =: J_1 + J_2 + J_3 + J_4. \end{aligned} \quad (36)$$

Further, since

$$\int_\Omega W_0^\varepsilon(\varepsilon \nabla v_\varepsilon) \chi_0^\varepsilon dx = \int_{\Omega \times Y_0} W_0^\varepsilon(\nabla_y \tilde{v}_\varepsilon + \nabla_y \psi^\varepsilon) dy dx,$$

and due to (5) we can estimate the first term in (36) as follows:

$$\begin{aligned} |J_1| &\leq \sum_{s,t} \int_{K_{st}} |W_0^\varepsilon(\nabla_y \tilde{v}_\varepsilon + \nabla_y \psi^\varepsilon) - W_0^\varepsilon(e_{s,t}^\varepsilon + \nabla_y \psi^\varepsilon)| \, dy \, dx \\ &\leq \sum_{s,t} \int_{K_{st}} \beta \left(1 + |\nabla_y \tilde{v}_\varepsilon + \nabla_y \psi^\varepsilon|^{p-1} + |e_{s,t}^\varepsilon + \nabla_y \psi^\varepsilon|^{p-1} \right) |\nabla_y \tilde{v}_\varepsilon - e_{s,t}^\varepsilon| \, dy \, dx. \end{aligned} \quad (37)$$

Hence

$$J_1 \rightarrow 0$$

as $\varepsilon \rightarrow 0$, due to boundedness of $\nabla_y \tilde{v}_\varepsilon$, $\nabla_y \psi^\varepsilon$, $e_{s,t}^\varepsilon$ in L^p and the convergence properties (29), (31). The second term J_2 in (36) is already estimated, see (35). The third term J_3 can be shown to converge to zero analogously to (37). Finally, $J_4 \rightarrow 0$ due to the pointwise convergence of QW_0^ε to QW_0 . Hence (30) holds true.

In the general case of $v(x, y) \in L^p(\Omega, W_0^{1,p}(Y_0))$ we approach v by a sequence of C^∞ functions: $v_i \rightarrow v$ and $\nabla_y v_i \rightarrow \nabla_y v$ in $L^p(\Omega \times Y_0)$. For each v_i we can apply the lemma as shown above. Then using the continuity property of the integral functional we can find the required sequence by the method of extraction of a diagonal subsequence via Proposition 17. \square

Combining the results of Lemma 21 and Lemma 22 we arrive at the following statement.

Theorem 23 *Suppose that the family of stored-energy functions W_0^ε , $\varepsilon > 0$, satisfies the conditions of Section 2. Then the two-scale Γ -limit of the functional I_0^ε (see (24)) is given by*

$$I_0^{\text{hom}}(v) = \Gamma_{\text{ts}}\text{-}\lim_{\varepsilon \rightarrow 0} I_0^\varepsilon(v) = \int_{\Omega} \int_{Y_0} QW_0(\nabla_y v) \, dy \, dx. \quad (38)$$

8 Splitting the functional into the stiff and soft components

In previous sections we studied two-scale Γ -limits of two integral functionals, I_1^ε and I_0^ε , corresponding to the stiff and soft components of the elastic body. In what follows we demonstrate that the full functional I^ε can be presented as a sum of I_1^ε and I_0^ε and an asymptotically small term. Consequently, the two-scale Γ -limit of I^ε can be obtained as a sum of the two-scale Γ -limits of I_1^ε and I_0^ε .

Consider the integral functional I^ε (cf. (7)) with the potential energy of applied forces included:

$$I^\varepsilon(u) = \int_{\Omega} (W_1(\nabla u) \chi_1^\varepsilon + W_0^\varepsilon(\varepsilon \nabla u) \chi_0^\varepsilon) \, dx - \int_{\Omega} f \cdot u \, dx. \quad (39)$$

Let a sequence u_ε be such that $I^\varepsilon(u_\varepsilon) \leq C$. Then $\|\nabla u_\varepsilon \chi_1^\varepsilon\|_{L^p(\Omega)} \leq C$, $\|\varepsilon \nabla u_\varepsilon \chi_0^\varepsilon\|_{L^p(\Omega)} \leq C$ and hence the following representation is valid:

$$u_\varepsilon = \tilde{u}_\varepsilon + v_\varepsilon,$$

where $\|\nabla \tilde{u}_\varepsilon\|_{L^p(\Omega)} \leq C$ and $v_\varepsilon \in W_0^{1,p}(E_0^\varepsilon)$. (Note the continuous extension Lemma given in Section 3.) We split the value of the functional I^ε on u_ε into a sum of two

terms involving functionals I_1^ε and I_0^ε (cf. (22)) that correspond to the stiff and soft components and some small term (as $\varepsilon \rightarrow 0$):

$$I^\varepsilon(u_\varepsilon) = \widehat{I}_1^\varepsilon(\tilde{u}_\varepsilon) + \widehat{I}_0^\varepsilon(v_\varepsilon) + R^\varepsilon(u_\varepsilon), \quad (40)$$

where

$$\begin{aligned} \widehat{I}_1^\varepsilon(\tilde{u}_\varepsilon) &= \int_{\Omega} W_1(\nabla \tilde{u}_\varepsilon) \chi_1^\varepsilon \, dx - \int_{\Omega} f \cdot \tilde{u}_\varepsilon \, dx, \\ \widehat{I}_0^\varepsilon(v_\varepsilon) &= \int_{\Omega} W_0^\varepsilon(\varepsilon \nabla v_\varepsilon) \chi_0^\varepsilon \, dx - \int_{\Omega} f \cdot v_\varepsilon \, dx, \\ R^\varepsilon(u_\varepsilon) &= \int_{\Omega} (W_0^\varepsilon(\varepsilon \nabla u_\varepsilon) - W_0^\varepsilon(\varepsilon \nabla v_\varepsilon)) \chi_0^\varepsilon \, dx. \end{aligned} \quad (41)$$

Note that the functionals $\widehat{I}_1^\varepsilon, \widehat{I}_0^\varepsilon$ differ from $I_1^\varepsilon, I_0^\varepsilon$ only by a linear term due to the applied forces density f . Condition (5) yields

$$|R^\varepsilon(u_\varepsilon)| \leq C \int_{\Omega} (1 + |\varepsilon \nabla u_\varepsilon|^{p-1} + |\varepsilon \nabla v_\varepsilon|^{p-1}) |\varepsilon \nabla \tilde{u}_\varepsilon| \chi_0^\varepsilon \, dx.$$

Then, via the Hölder inequality,

$$R^\varepsilon(u_\varepsilon) \rightarrow 0 \quad (42)$$

as $\varepsilon \rightarrow 0$. This immediately proves that

$$\liminf_{\varepsilon \rightarrow 0} \inf_u I^\varepsilon(u) = \liminf_{\varepsilon \rightarrow 0} \inf_{\tilde{u}} \widehat{I}_1^\varepsilon(\tilde{u}) + \liminf_{\varepsilon \rightarrow 0} \inf_v \widehat{I}_0^\varepsilon(v).$$

In other words one can deal with the functionals $\widehat{I}_1^\varepsilon$ and $\widehat{I}_0^\varepsilon$ separately in the context of minimising sequences. We will employ this observation in the argument of Section 10.

In addition, as was shown in Section 5, if $u_\varepsilon \xrightarrow{2} u(x, y)$ then either $\liminf_{\varepsilon \rightarrow 0} I^\varepsilon(u_\varepsilon) = +\infty$ or $u(x, y) = u_0(x) + v(x, y)$, where $u_0 \in W^{1,p}(\Omega)$ and $v \in L^p(\Omega, W_0^{1,p}(Y_0))$. In the latter case one has, in view of (40), (42),

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} I^\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon \rightarrow 0} \widehat{I}_1^\varepsilon(\tilde{u}_\varepsilon) + \liminf_{\varepsilon \rightarrow 0} \widehat{I}_0^\varepsilon(v_\varepsilon) \\ &\geq I_1^{\text{hom}}(u_0) + I_0^{\text{hom}}(v) - \int_{\Omega} \int_{Y_0} f \cdot (u_0 + v) \, dy \, dx, \end{aligned}$$

where $I_1^{\text{hom}}, I_0^{\text{hom}}$ are defined by (23), (38).

Further, suppose that $u_0 \in W^{1,p}(\Omega)$, $v \in L^p(\Omega, W_0^{1,p}(Y_0))$ are given. Let v_ε be the sequence from Lemma 22, and \tilde{u}_ε be such that $\tilde{u}_\varepsilon \rightarrow u_0$ in $L^p(\Omega)$ as $\varepsilon \rightarrow 0$ and $\lim_{\varepsilon \rightarrow 0} I_1^\varepsilon(\tilde{u}_\varepsilon) = I_1^{\text{hom}}(u_0)$, see Theorem 19. Then, clearly, $u_\varepsilon := \tilde{u}_\varepsilon + v_\varepsilon \xrightarrow{2} u_0 + v$ and (cf. (40), (42))

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I^\varepsilon(u_\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \widehat{I}_1^\varepsilon(\tilde{u}_\varepsilon) + \lim_{\varepsilon \rightarrow 0} \widehat{I}_0^\varepsilon(v_\varepsilon) + \lim_{\varepsilon \rightarrow 0} R^\varepsilon(u_\varepsilon) \\ &= I_1^{\text{hom}}(u_0) + I_0^{\text{hom}}(v) - \int_{\Omega} \int_{Y_0} f \cdot (u_0 + v) \, dy \, dx. \end{aligned}$$

Hence, we have proved the following theorem.

Theorem 24 *The two-scale Γ -limit of the functional I^ε , see (39), is given by*

$$I^{\text{hom}} := \Gamma_{\text{ts}}\text{-}\lim_{\varepsilon \rightarrow 0} I^\varepsilon(u) = \begin{cases} I_1^{\text{hom}}(u_0) + I_0^{\text{hom}}(v) - \int_{\Omega} \int_{Y_0} f \cdot u \, dy \, dx, \\ \quad \text{if } u = u_0 + v \in W^{1,p}(\Omega) + L^p(\Omega, W_0^{1,p}(Y_0)), \\ +\infty, \text{ otherwise.} \end{cases} \quad (43)$$

While the above two-scale Γ -convergence statement fully characterises the two-scale behaviour as $\varepsilon \rightarrow 0$ of the functionals I^ε given by (1), it is worth pointing out that at the moment there is no statement available about *a priori* compactness of minimising sequences for I^ε , and so the two-scale analogue of Theorem 5 does not apply. This implies that: 1) There is no automatic attainability of the infimum of the limit functional I^{hom} ; 2) As far as the infima of I^ε are concerned, only the inequality $\limsup_{\varepsilon \rightarrow 0} \inf I^\varepsilon \leq \inf I^{\text{hom}}$ is guaranteed, and further study is needed regarding the convergence of $\inf I^\varepsilon$ to $\inf I^{\text{hom}}$ as $\varepsilon \rightarrow 0$. In the next two sections we demonstrate that both these issues are resolved positively for the functionals (1).

9 Existence of minimum of the limit functional

In this section we prove that the limit functional I^{hom} attains its minimum. Consider the part of the limit functional corresponding to the soft component (*cf.* (38))

$$\tilde{I}_0^{\text{hom}}(v) = \int_{\Omega} \int_{Y_0} QW_0(\nabla_y v) \, dy \, dx - \int_{\Omega} \int_{Y_0} f \cdot v \, dy \, dx,$$

defined on $v \in L^p(\Omega, W_0^{1,p}(Y_0))$. Since f depends only on x , we can consider x as a parameter in the problem of minimisation of the functional. Thus we consider the minimisation of

$$F_t(w) = \int_{Y_0} QW_0(\nabla w) \, dy - t \cdot \int_{Y_0} w \, dy,$$

where $w \in W_0^{1,p}(Y_0)$ and $t \in \mathbb{R}^m$ represents $f(x)$. It is well known that for any t this functional attains its minimum. Let us denote by $\mathscr{W}(t)$ the set of all minimisers of F_t . Thus we are dealing with a set-valued function defined on \mathbb{R}^m with values being subsets of $[L^p(Y_0)]^m$, $\mathscr{W}(t) \subset [L^p(Y_0)]^m$. We need to show the existence of a selection $w : \mathbb{R}^m \rightarrow [L^p(Y_0)]^m$, $w(t) \in \mathscr{W}(t)$, which is measurable as a function of two variables $(t, y) \in \mathbb{R}^m \times Y_0$. To this end we will employ the measurable selection theorem, but first we would like to introduce a few definitions.

Definition 25 Let X be a topological space, $\mathscr{B}(X)$ be the σ -algebra of Borel sets of X , $f : \mathbb{R}^m \rightarrow X$, $F : \mathbb{R}^m \rightarrow \{A : A \subset X\}$. For $A \subset X$ we define its inverse image $F^-(A) = \{t : F(t) \cap A \neq \emptyset\}$. We say that f is measurable if $f^{-1}(B)$ is measurable for $B \in \mathscr{B}(X)$. We say that F is weakly measurable (as a set-valued map) if $F^-(B)$ is measurable for all open $B \subset X$. A map $f : \mathbb{R}^m \rightarrow X$ is called simple if it has finitely many values.

Let us show that \mathscr{W} defined above is weakly measurable as a set-valued function. The topology in the space $L^p(Y_0)$ is determined by the underlying norm. Let $B \subset L^p(Y_0)$ be closed. Let $\{t_n\} \subset \mathscr{W}^-(B)$ be an arbitrary converging sequence, $t_n \rightarrow t$. For every n there exists a minimiser w_n of F_{t_n} which belongs to B . The sequence w_n is bounded in $W^{1,p}(Y_0)$ as it follows from the standard growth conditions for QW_0 . Then there exists a converging strongly in $L^p(Y_0)$ and weakly in $W^{1,p}(Y_0)$ subsequence $w_n \rightarrow w$ (not relabelled). It is easy to see that the sequence F_{t_n} Γ -converges to F_t as $n \rightarrow \infty$, where the Γ -convergence is understood with respect to the convergence in $L^p(Y_0)$ (see Definition 3). Indeed, the lower semicontinuity type inequality follows from quasiconvexity of the integrand QW_0 and the fact that the integral functionals with quasiconvex integrand are lower semicontinuous with respect to weak convergence in $W^{1,p}(Y_0)$. Existence of the recovery sequence is trivial. So by Theorem 5 the limit function w is a minimiser of F_t , and, hence $w \in \mathscr{W}(t)$. On the other hand $w \in B$ since B is closed set. We conclude that $t \in \mathscr{W}^-(B)$, that is $\mathscr{W}^-(B)$ is closed set. Any open set A in $L^p(Y_0)$ can be presented as a countable union of closed sets $A = \cup_i B_i$. Hence its inverse image $\mathscr{W}^-(A)$ is measurable as a countable union of closed sets $\mathscr{W}^-(B_i)$. This proves that the map \mathscr{W} is weakly measurable.

By the well-known measurable selection theorem (see *e.g.* [36, Theorem 3.1], [20]) \mathscr{W} has a selection, which is measurable (in the sense of Definition 25), and we denote this selection by $w = w(t)$. By the well-known measurability criterion (see *e.g.* [16, Proposition E.2]) there exists a sequence w_n of measurable simple maps such that

$$w(t) = \lim_{n \rightarrow \infty} w_n(t) \quad (44)$$

and

$$\|w_n(t)\|_{L^p(Y_0)} \leq \|w(t)\|_{L^p(Y_0)}. \quad (45)$$

The relations (44) and (45) hold for all t and the limit in (44) is understood with respect to the convergence in $L^p(Y_0)$ -norm. The key point here is that every simple map w_n is Lebesgue measurable as a function of two variables $w_n(t, y) : \mathbb{R}^m \times Y_0 \rightarrow \mathbb{R}^m$. Using the standard growth conditions and the Poincaré inequality it is easy to obtain the following estimate for $w(t)$ (as a minimiser of F_t):

$$\|w(t)\|_{L^p(Y_0)}^p \leq \|w(t)\|_{W^{1,p}(Y_0)}^p \leq C(|t|^q + 1). \quad (46)$$

Indeed,

$$\begin{aligned} \|\nabla w(t)\|_{L^p(Y_0)}^p &\leq \int_{Y_0} QW_0(\nabla w(t)) \, dy = \min F_t + t \cdot \int_{Y_0} w \, dy \\ &\leq C(1 + |t| \|w(t)\|_{L^p(Y_0)}) \leq C(1 + |t| \|\nabla w(t)\|_{L^p(Y_0)}), \end{aligned}$$

from which (46) follows.

Then from (45) and (46) it follows that $w_n(t, y) \in L^p(I \times Y_0)$ for any bounded set $I \subset \mathbb{R}^m$. Moreover, from (44) we see that w_n is a Cauchy sequence in $L^p(I \times Y_0)$, and hence its limit in $L^p(I \times Y_0)$ coincides with w . Hence we conclude that the mapping w is measurable as a function of two variables $w(t, y)$ and belongs to $L_{\text{loc}}^p(\mathbb{R}^m, W_0^{1,p}(Y_0))$.

Substituting t with $f(x)$ we obtain $v(x, y) = w(f(x), y)$, $v \in L^p(\Omega, W_0^{1,p}(Y_0))$, cf. (46). Obviously v is a minimiser of I_0^{hom} .⁹

10 Convergence of infima of the functionals I^ε

Next, as was outlined at the end of Section 8, we prove that

$$\liminf_{\varepsilon \rightarrow 0} \inf_u I^\varepsilon(u) \geq \min_u I^{\text{hom}}(u), \quad (47)$$

where (cf. (43))

$$\begin{aligned} I^{\text{hom}}(u) &= \int_{\Omega} W_1^{\text{hom}}(\nabla u_0) \, dx + \int_{\Omega} \int_{Y_0} QW_0(\nabla_y v) \, dy \, dx - \int_{\Omega} \int_Y f \cdot u \, dy \, dx, \\ W_1^{\text{hom}}(e) &= \liminf_{t \rightarrow \infty} \left\{ \frac{1}{t^n} \int_{(0,t)^n \cap E_1} W_1(e + \nabla \varphi) \, dx : \varphi \in W_0^{1,p}((0,t)^n) \right\}, \\ u &= u_0(x) + v(x, y) \in W^{1,p}(\Omega) + L^p(\Omega, W_0^{1,p}(Y_0)). \end{aligned} \quad (48)$$

In combination with the inequality $\limsup_{\varepsilon \rightarrow 0} \inf I^\varepsilon \leq \min I^{\text{hom}}$ (which is a consequence of (43)), this will imply the equality $\lim_{\varepsilon \rightarrow 0} \inf I^\varepsilon = \min I^{\text{hom}}$.

Aiming to get a contradiction with (47) we assume the contrary, *i.e.*

$$\liminf_{\varepsilon \rightarrow 0} \inf_u I^\varepsilon(u) < \min_u I^{\text{hom}}(u).$$

Then, by the virtue of the procedure described in Section 8, there exists a sequence $u_\varepsilon = \tilde{u}_\varepsilon + v_\varepsilon$ such that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I^\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon \rightarrow 0} \inf_u I^\varepsilon(u) \\ &= \liminf_{\varepsilon \rightarrow 0} \inf_{\tilde{u}} \widehat{I}_1^\varepsilon(\tilde{u}) + \liminf_{\varepsilon \rightarrow 0} \inf_v \widehat{I}_0^\varepsilon(v) < \min_u I^{\text{hom}}(u), \end{aligned} \quad (49)$$

where $\widehat{I}_1^\varepsilon$ and $\widehat{I}_0^\varepsilon$ are as defined in (41), and $v_\varepsilon \xrightarrow{2} v(x, y)$, $\varepsilon \nabla v_\varepsilon \xrightarrow{2} \nabla_y v(x, y)$, $\tilde{u}_\varepsilon \rightarrow u_0$ strongly in $L^p(\Omega)$ and weakly in $W^{1,p}(\Omega)$ for some u_0 and v as in (48). Notice that, by the construction of Section 8, \tilde{u}_ε and v_ε are infimising sequences for $\widehat{I}_1^\varepsilon$ and $\widehat{I}_0^\varepsilon$, respectively. It is well known from the classical theory of Γ -convergence, see [8], that

$$\begin{aligned} \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \widehat{I}_1^\varepsilon(\psi) &= I_1^{\text{hom}}(\psi) - \int_{\Omega} f \cdot \psi \, dy \, dx \\ &= \int_{\Omega} W_1^{\text{hom}}(\nabla \psi) \, dx - \int_{\Omega} f \cdot \psi \, dy \, dx, \quad \psi \in W^{1,p}(\Omega). \end{aligned}$$

Moreover,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \widehat{I}_1^\varepsilon(\tilde{u}_\varepsilon) &= \liminf_{\varepsilon \rightarrow 0} \inf_u \widehat{I}_1^\varepsilon(u) = \min_u \left(I_1^{\text{hom}}(u) - \int_{\Omega} f \cdot u \, dy \, dx \right) \\ &= I_1^{\text{hom}}(u_0) - \int_{\Omega} f \cdot u_0 \, dy \, dx. \end{aligned} \quad (50)$$

⁹ Perhaps a person more familiar with measure theory might suggest a shorter proof of the existence of a measurable minimiser or even claim that this fact is ‘‘obvious’’.

Further, denote the unfolding of v_ε by \widehat{v}_ε , i.e. $\widehat{v}_\varepsilon(x, y) = \mathcal{T}_\varepsilon(v_\varepsilon)(x, y)$ (see Definition 13). Since $v_\varepsilon \xrightarrow{2} v$ and hence $\widehat{v}_\varepsilon \rightharpoonup v$ in $L^p(\Omega)$, one has

$$\lim_{\varepsilon \rightarrow 0} \left\{ \int_{\Omega} f \cdot v_\varepsilon \, dx - \int_{\Omega} \int_{Y_0} f \cdot \widehat{v}_\varepsilon \, dy \, dx \right\} = 0. \quad (51)$$

Notice also, that $\widehat{v}_\varepsilon \in L^p(\Omega, W_0^{1,p}(Y_0))$, and

$$\int_{\Omega} W_0^\varepsilon(\nabla v_\varepsilon) \chi_0^\varepsilon \, dx = \int_{\Omega} \int_{Y_0} W_0^\varepsilon(\nabla_y \widehat{v}_\varepsilon) \, dy \, dx \geq \int_{\Omega} \int_{Y_0} QW_0^\varepsilon(\nabla_y \widehat{v}_\varepsilon) \, dy \, dx. \quad (52)$$

Clearly, from (43), (49), (50), (51) and (52) it follows that for sufficiently small ε one has

$$I^{\text{hom}}(u_0 + \widehat{v}_\varepsilon) < \min_u I^{\text{hom}},$$

which is a contradiction. Hence (47) is valid. This, together with Theorem 24, implies the following statement

Theorem 26 *For the functionals I^ε defined by (1) under the conditions of Section 2, one has*

$$\liminf_{\varepsilon \rightarrow 0} \min_u I^\varepsilon(u) = \min_u I^{\text{hom}}(u),$$

where I^{hom} is defined by (43). For any minimiser u of I^{hom} there exists a sequence u_ε such that $u_\varepsilon \xrightarrow{2} u$ and

$$\lim_{\varepsilon \rightarrow 0} I^\varepsilon(u_\varepsilon) = I^{\text{hom}}(u).$$

11 Concluding remarks

Here we offer a discussion of our results hoping to place them better in the context of the existing literature, and suggest possible avenues for further development.

It may be worth highlighting first the synergy of several known analytical methods within our approach, notably the two-scale convergence, periodic unfolding method and the concept of Γ -convergence. There is a number of previous works where these methods have been used in treating what we would like to refer to as “classical” models, whereby the macroscopic, homogenised, solution depends only on the slow variable. Importantly however, these approaches have a multi-scale character in that *a priori* they may assume a two-scale nature of the leading order term in the behaviour of the solutions as $\varepsilon \rightarrow 0$. For example, the two-scale convergence, either on its own or in combination with the procedure of unfolding, leads in the limit to functions that in general depend both on the slow and fast variable, and it is only by virtue of the standard $W^{1,p}$ -compactness that the single-scale structure of the limit is inferred *a posteriori*. The Γ -convergence method can then be viewed as a complementary tool that allows to deal with problems that may not have a weak formulation, in particular problems with non-convex energies. From this perspective our work fills an existing gap in the mathematical theory of homogenisation by demonstrating how the method of Γ -convergence can be used to analyse problems where the two-scale structure of the solution persists “to the very limit” as $\varepsilon \rightarrow 0$. A basic example of such a

setting is that of high-contrast composites, which fail to satisfy the property of uniform coercivity as $\varepsilon \rightarrow 0$, and hence do not possess the usual $W^{1,p}$ -compactness property. Prior to our work such problems had only been studied within settings that are amenable to direct treatment via the definition of the two-scale convergence (confusingly enough, this would correspond to what is referred to as the indirect approach in the calculus of variations). In view of the fundamental importance of those physical settings that are underpinned by some kind of energy minimisation principle, as *e.g.* the setting of static non-linear elasticity, one is led to asking whether “non-classical” non-convex problems, in particular those that exhibit high-contrast in the constitutive response at different material points, can be analysed by combining the two-scale and variational approaches. Thus one arrives at the concept of “two-scale Γ -convergence”, which we define in Section 5 and which exploits two-scale, rather than one-scale, compactness properties of variational integrals.

It is perhaps appropriate to note here that as an abstract concept, the two-scale Γ -convergence is not new: some elements of it have appeared, *e.g.*, in the works [34], [35], [21]. The rationale leading up to the related theory becomes transparent when one notices that the usual L^p -norm convergence and the strong two-scale convergence are equivalent under the unfolding transformation, and hence the “two-scale” Γ -convergence can be defined using the standard Γ -convergence in two-variable L^p spaces. While this approach yields the desired effective material description and is a convenient tool of studying solutions to those problems where lack of compactness is not an issue, in the non-classical settings a more direct approach may be preferable where one deals with sequences of functions in the original function space ($L^p(\Omega)$ in our case) rather than those that are the result of unfolding. Convergence statements for oscillating sequences of a given two-scale functional are then formulated in terms of the sequences themselves and not in terms their “unfolded” counterparts whose limits would require “folding back” to the original variables.

The technique presented in this paper is aimed at treating high-contrast functionals of the “double-porosity” type, in that the underlying periodic composite is made of two types of materials, “stiff” and “soft”, whose individual constitutive relations exhibit a high degree (of order ε) of contrast between each other and which occupy certain domains within the unit cell. While the individual stored-energy function for each of these two contrasting material types may vary “moderately” over the unit cell leaving the mathematical statements within our work essentially unchanged, large (*i.e.* tending to infinity as $\varepsilon \rightarrow 0$) variations within each type are not covered by our theory. This restriction is manifested by the fact that the stored-energy functions $W^\varepsilon = W^\varepsilon(y, e)$ for the composite family in question are postulated to have the form $W^\varepsilon(y, e) = W_0(y, e)\chi_0(y) + W_1(y, \varepsilon e)\chi_1(y)$, which corresponds to the strict separation between the “stiff” and “soft” materials. Another limitation of our results is that at present they only apply to materials that are either isotropic or at worst weakly anisotropic at each point in space, *i.e.* the contrast in their constitutive response in different spatial directions does not get large as $\varepsilon \rightarrow 0$. Extending the two-scale Γ -convergence to highly anisotropic (or “partially degenerate”) materials (see the works [12], [30] in the linearised setting) would be an important step forward, in our opinion, both from the mathematical and the applied perspective, in view of their potential for applications

as non-linear metamaterials. The key difficulty in doing so within the present framework is posed by the inability to consider the stiff and soft parts of the related integral functional separately from each other when treating such partially degenerate situations.

From the point of view of the mathematical theory of non-linear elasticity, there is one significant drawback of our technique as described, namely the fact that the growth condition (3) is incompatible with the requirement that $W^\varepsilon(y, e) \rightarrow \infty$ as $\det(e) \rightarrow 0+$. The associated technical difficulties, however, are present even in the “moderate contrast” setting of the standard mathematical elasticity, see [2]. In particular, even the issues of the existence of minimisers and the validity of the weak Euler-Lagrange equation are known to be open for non-polyconvex functionals of this kind even in the absence of periodic microstructure. Hence, in our work we feel justified to restrict ourselves to the case of the usual global growth condition (3), awaiting further progress on these problems elsewhere. It should perhaps be mentioned, however, that the theory we propose allows for minimising sequences that are “discontinuous at the microscale”: an example of these is obtained by taking *e.g.* a cavitating minimiser of the “soft” part of the limiting functional. This feature seems to be inherently out of reach by the “classical” theories.

One general analytical issue that is raised by our theory is whether one can strengthen Lemma 21 (Section 7.1) by proving the inequality $I^{\text{hom}}(u) \leq \liminf_{\varepsilon \rightarrow 0} I^\varepsilon(u_\varepsilon)$ with respect to the weak two-scale convergence. A simple example showing that the answer to this question is negative, is via a “checkerboard” sequence u_ε constructed by taking a functional I^{hom} whose soft component admits two different global minimisers ψ_1 and ψ_2 , setting $u_\varepsilon = \psi_i(x/\varepsilon)$ within the soft part of each periodicity cell of size ε so that the values of the index i in any two neighbouring cells are different, and extending u_ε to zero elsewhere. The possible non-compactness of a minimising sequence for I^ε is illustrated using the same example: clearly the sequence u_ε weakly two-scale converges to $\chi_0(y)(\psi_1(y) + \psi_2(y))/2$ but it fails to converge in the strong two-scale sense. Clearly, in the case when the stored-energy function W_0 is convex, this kind of counterexample is not possible, and for such functionals we expect a two-scale analogue the Mosco convergence to hold (see *e.g.* [23])

Our technique calls for an extension in various directions. In addition to the case of partially degenerate composites already mentioned, one could consider classes of stored-energy functions with different growth exponents for the “stiff” and “soft” components, as well as combinations of high-contrast periodic with “thin” structures, where the volume fraction of the stiff component vanishes in the limit as $\varepsilon \rightarrow 0$. The technique of “ Γ -developments”, which can be thought of as an analogue of the method of asymptotic expansions in the non-linear setting (see [10] and references therein), should also be amenable to an extension using the two-scale Γ -convergence approach presented here, to the contexts where the two-scale behaviour persists to the limit as $\varepsilon \rightarrow 0$. We plan to address these issues elsewhere.

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