Homotopical Algebra for C^*-algebras

The 3rd Annual EU-NCG conference,
June 28 - July 2, 2010

Otgonbayar Uuye
University of Copenhagen, Denmark
otogo@math.ku.dk

July 1, 2010

Homotopical Algebra: Model Category

Pointed Model Category $\mathbf{M}$

- weak equivalences
- fibrations
- cofibrations

gives

- homotopy category $\mathbf{Ho}(\mathbf{M})$
- loop and suspension adjunction

$$\Omega : \mathbf{Ho}(\mathbf{M}) \to \mathbf{Ho}(\mathbf{M}) : \Sigma$$

- triangulated categories

$$\mathbf{Ho}(\mathbf{M})[\Omega^{-1}] \quad \text{and} \quad \mathbf{Ho}(\mathbf{M})[\Sigma^{-1}]$$

Homotopical Algebra: Category of Fibrant Objects

Pointed Category of Fibrant Objects $\mathbf{C}$

- weak equivalences
- fibrations
- cofibrations

gives

- homotopy category $\mathbf{Ho}(\mathbf{C})$
- loop and suspension adjunction

$$\Omega : \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C}) : \Sigma$$

- triangulated category

$$\mathbf{Ho}(\mathbf{C})[\Omega^{-1}] \quad \text{and} \quad \mathbf{Ho}(\mathbf{C})[\Sigma^{-1}]$$

Category of Fibrant Objects: Weak Equivalences

Definition

C = category
W = weak equivalences = $\xrightarrow{\sim}$

(W1) isomorphisms are weak equivalences

(W2) if 2 of f, g and fg are weak equivalences, then so is the third

Homotopy category of **C** is $\mathbf{Ho}(\mathbf{C}) = \mathbf{C}[\mathbf{W}^{-1}]$.

Example

C = **Top**

A map $t : A \to B$ is π_0-*equivalence* iff

$$\pi_0(t) : \pi_0(A) \to \pi_0(B)$$

bijection.

Category of Fibrant Objects: Fibrations

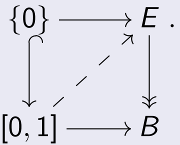
Definition

C = category with terminal object = $*$
F = fibrations = $\twoheadrightarrow$

- (F0) fibrations are closed under composition
- (F1) isomorphisms are fibrations
- (F2) pull-back of a fibration is a fibration
- (F3) $B \to *$ is a fibration for all B

Example

C = **Top**, $*$ = one-point set
A map $p : E \to B$ is π_0-*fibration* iff it has path-lifting property:



Category of Fibrant Objects: Acyclic Fibrations

Definition

$\mathbf{C}$ = category of fibrant objects with terminal object $= *$

- $\mathbf{W}$ = weak equivalences
- $\mathbf{F}$ = fibrations
- $\mathbf{F} \cap \mathbf{W}$ = acyclic fibration $= \overset{\sim}{\twoheadrightarrow}$

(FW1) pull-back of an acyclic fibration is an acyclic fibration

(FW2) path-objects exist: the diagonal map $B \to B \times B$ factors as

$$B \xrightarrow{\sim} B^I \twoheadrightarrow B \times B$$

$\mathbf{C}$ is *pointed* if $*$ is also initial object

Category of Fibrant Objects: Example

$\mathbf{C} = \mathbf{Top}$, $* =$ one-point space

- $\mathbf{W} = \pi_0$-equivalences
- $\mathbf{F} = \pi_0$-fibrations

Lemma

If $p : E \to B$ is in $\mathbf{F} \cap \mathbf{W}$, then p is surjective.

Proof.

Let $b \in B$. Then there is a diagram

$$\begin{array}{ccc} \{0\} & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ [0,1] & \xrightarrow{\;h\;} & B \end{array}$$

with $h(1) = b$. Lifting h to a path in E, and evaluating at 1, we get $e \in E$ such that $p(e) = b$. $\square$

Ordinary Homotopy Theory of C^*-algebras

$\mathbf{C}^*$ = category of C^*-algebras and $*$-homomorphisms,

- $\prod$ = product of algebras, finite product = direct sum
- $\coprod$ = free product of algebras, usually denoted $*$
- enriched over **Top**:

$$\mathbf{Top}(X, \mathbf{C}^*(A, B)) \cong \mathbf{C}^*(A, B \otimes C(X))$$

Definition

$\pi_0\mathbf{C}^*$ = homotopy category of $\mathbf{C}^*$:

$$\pi_0\mathbf{C}^*(A, B) := \pi_0\mathbf{C}^*(A, B)$$

Want category of fibrant objects on $\mathbf{C}^*$ such that

$$\mathbf{Ho}(\mathbf{C}^*) = \pi_0\mathbf{C}^*$$

Category of Fibrant Objects on $\mathbf{C}^*$ I

Definition

$\mathbf{C} = \mathbf{C}^*$, pointed $* = 0$
$\mathbf{W}$ = homotopy equivalences = $t : A \to B$ such that

$$t_* : \mathbf{C}^*(D, A) \to \mathbf{C}^*(D, B)$$

is π_0-equivalence for all D
$\mathbf{F}$ = Schochet cofibrations = $p : E \to B$ such that

$$p_* : \mathbf{C}^*(D, E) \to \mathbf{C}^*(D, B)$$

is π_0-fibration for all D

Category of Fibrant Objects on $\mathbf{C}^*$ II

Proposition

This makes $\mathbf{C}^*$ into category of fibrant objects, with homotopy category $\pi_0\mathbf{C}^*$ and loop-object $\Omega B = B \otimes C_0(0,1)$.

Corollary

The category

$$\mathbf{SW}(\mathbf{C}^*) := \mathbf{Ho}(\mathbf{C}^*)[\Omega^{-1}]$$

with objects (A,n), $A \in \mathbf{C}^*$, $n \in \mathbb{Z}$ and morphisms

$$\mathbf{SW}(\mathbf{C}^*)((A,n),(B,m)) := \operatorname*{colim}_{k\to\infty} \mathbf{Ho}(\mathbf{C}^*)(\Omega^{n+k}A, \Omega^{m+k}B)$$

is a triangulated category with shift $\Sigma = \Omega^{-1}$ and distinguished triangles

$$(\Omega B, n) \dashrightarrow (F,n) \rightarrowtail (E,n) \twoheadrightarrow (B,n)$$

K-theory

Definition

$t : A \to B$ is a K-equivalence if $t_* : K_*(A) \to K_*(B)$ is an isomorphism.

Proposition

K-equivalences and Shochet cofibrations define a category of fibrant objects, denoted $\mathbf{K}$. *Then* $\mathbf{Ho}(\mathbf{K})$ *is a triangulated category and for* $B \in \mathbf{K}$, *we have*

$$\mathbf{Ho}(\mathbf{K})(\mathbb{C}, B) \cong K_0(B).$$

More generally, for A, $B \in \mathbf{K}$, *there is a natural short exact sequence*

$$\mathrm{Ext}(K(A), K(B)) \rightarrowtail \mathbf{Ho}(\mathbf{K})(A, B) \twoheadrightarrow \mathrm{Hom}(K(A), K(B)) .$$

Proof of UCT: Surjectivity

Enough to show natural map

$$\mathbf{Ho}(\mathbf{K})(A, B) \to \mathrm{Hom}(K(A), K(B)) \tag{1}$$

is an isomorphism if $K_*(A)$ is *free*. First note

$$K_0(D) = \mathbf{Ho}(\mathbf{C}^*)(q\mathbb{C}, D \otimes \mathcal{K}), \tag{2}$$
$$K_1(D) = \mathbf{Ho}(\mathbf{C}^*)(\Omega\mathbb{C}, D \otimes \mathcal{K}). \tag{3}$$

Surjectivity:

$$\begin{array}{ccc} (\coprod_I q\mathbb{C}) \coprod (\coprod_J \Omega\mathbb{C}) & \longrightarrow & B \otimes \mathcal{K} \\ \downarrow \wr & & \uparrow \\ A \otimes \mathcal{K} & & \wr \\ \uparrow \wr & & \\ A & & B \end{array} \tag{4}$$

Proof of UCT: Injectivity

Injectivity:

$$\begin{array}{ccccc}
(\coprod_I q\mathbb{C}) \coprod (\coprod_J \Omega\mathbb{C}) & \xrightarrow{\sim} & A' \otimes \mathcal{K} & \longrightarrow & B \otimes \mathcal{K} \\
\downarrow \wr & & & & \\
A \otimes \mathcal{K} & & \uparrow \wr & & \uparrow \wr \\
\uparrow \wr & & & & \\
A & \xleftarrow{\sim} & A' & \longrightarrow & B
\end{array} \tag{5}$$

The general case follows using a geometric resolution of $K_*(A)$. □

C^*-stability

Definition

$t : A \to B$ is $\mathcal{K}$-homotopy equivalence iff

$$t \otimes \mathrm{id}_{\mathcal{K}} : A \otimes \mathcal{K} \to B \otimes \mathcal{K} \tag{6}$$

is homotopy equivalence.

Proposition

$\mathcal{K}$-homotopy equivalences and Schochet cofibrations give a category of fibrant objects, denoted $\mathbf{M}$*. For any* A*,* $B \in \mathbf{M}$*,*

$$\mathbf{Ho}(\mathbf{M})(A, B) \cong \mathbf{Ho}(\mathbf{C}^*)(A \otimes \mathcal{K}, B \otimes \mathcal{K}) \tag{7}$$

$$\cong \mathbf{Ho}(\mathbf{C}^*)(A, B \otimes \mathcal{K}). \tag{8}$$

KK-theory

Definition

$t : A \to B$ is a KK-equivalence iff

$$t_* : \mathbf{C}^*(qD, A \otimes \mathcal{K}) \to \mathbf{C}^*(qD, B \otimes \mathcal{K}) \tag{9}$$

is a π_0-equivalence for all D.

Proposition

KK-equivalences and Schochet cofibrations define a category of fibrant objects on separable C^*-algebras, denoted $\mathbf{KK}$*, whose homotopy category* $\mathbf{Ho}(\mathbf{KK})$ *is a triangulated category, equivalent to the category KK of Kasparov.*

E-theory

Definition

$t : A \to B$ is E-equivalence if it induces isomorphism on all half-exact, homotopy-invariant, C^*-invariant functors.

Proposition

E-equivalences and surjections define a category of fibrant objects on separable C^*-algebras, denoted $\mathbf{E}$, whose homotopy category $\mathbf{Ho}(\mathbf{E})$ is a triangulated category, equivalent to the category E of Higson.