

INSTANTONS ON NONCOMMUTATIVE TORIC VARIETIES

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Toric geometry

- ▶ **Definition:** A complex variety X is a **toric variety of dimension n** if it densely contains a (complex) algebraic torus $T = (\mathbb{C}^\times)^n$ and the natural T -action on itself (by translations) extends to X
- ▶ **Examples:** $X = T, \mathbb{C}^n, \mathbb{P}^n$
- ▶ X described by combinatorial data encoded in **toric diagram**:
 - ▶ **Vertices** $f =$ fixed points of T -action on X , with T -invariant open chart $\cong \mathbb{C}^n$
 - ▶ **Edges** $e = T$ -invariant projective lines \mathbb{P}^1 joining 2 fixed points f_1, f_2
 - ▶ **"Gluing rules"**: Near \mathbb{P}^1 , X looks like $\mathcal{O}_{\mathbb{P}^1}(-m_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-m_{n-1})$ – normal bundle determines local geometry of X near edge
- ▶ Or equivalently in dual **toric fan**: **Maximal (polyhedral) cones** dual to vertices define toric open cover of X ; $n - 1$ -cones dual to edges; **Gluing rules** along adjacent faces $\sigma \cap \tau$ of cones σ, τ

Motivation – Instanton counting

- ▶ Heuristic **local** enumeration of (generalized) noncommutative instantons on each $\mathbb{C}^n \subset X$ assemble to global quantities using gluing rules of toric geometry
(Nekrasov '02, . . . ; Iqbal *et al.* '08)
- ▶ Determines instanton contributions to SUSY gauge theory partition functions on X ; computes enumerative invariants of varieties **e.g.**, Seiberg–Witten and Donaldson invariants for $n = 2$, Donaldson–Thomas invariants for $n = 3$
- ▶ Find **global** notion of “noncommutative toric variety” and construction of instantons

Motivation – String geometry

- ▶ Chiral fermions on a “quantum curve” embed in string theory as intersecting D-branes with B -field; described by a holonomic D -module (Dijkgraaf, Hollands & Sulkowski '09)
- ▶ $\left\{ D\text{-modules} \right\} \iff \left\{ \text{modules on a noncommutative variety} \right\}$
- ▶ **Example:** $\left\{ \text{right ideals of Weyl algebra } \mathbb{C}[z, \partial_z] \right\}$
 $\iff \left\{ \text{line bundles on a noncommutative } \mathbb{P}^2 \right\}$
(Baranovsky, Ginzburg & Kuznetsov '02)
- ▶ Vector bundles on noncommutative \mathbb{P}^2
 \longleftrightarrow instantons on noncommutative \mathbb{R}^4
(Nekrasov & Schwarz '98; Kapustin, Kuznetsov & Orlov '01)

Classical instantons

- ▶ $SU(r)$ vector bundle $E \rightarrow M$ ($M = S^4$ or locally $M = \mathbb{R}^4$ and finite Yang–Mills energy) with anti-self-dual connection:

$$*F = -F$$

labelled by “topological charge” (instanton number)

$$c_2(E) = \frac{1}{8\pi^2} \int_M \text{Tr}(F \wedge F) = k \in H^4(M, \mathbb{Z})$$

- ▶ Rank r holomorphic vector bundles $E \rightarrow \mathbb{P}^2$ with $c_2(E) = k$, trivial on \mathbb{P}^1 at ∞ (Hitchin–Kobayashi correspondence)
- ▶ ADHM data (linear algebra!)
- ▶ Rank r holomorphic vector bundles $E \rightarrow \mathbb{P}^3$ with $c_2(E) = k$, trivial on \mathbb{P}^1 at ∞ , $H^1(\mathbb{P}^3, E(-2)) = 0$, reality condition (Atiyah–Penrose–Ward twistor correspondence)

Cocycle twist quantization

(Majid '95)

- ▶ H commutative Hopf algebra

$F : H \otimes H \longrightarrow \mathbb{C}$ convolution-invertible unital two-cocycle on H

- ▶ H_F – new Hopf algebra, $H = H_F$ as coalgebra, but with:

$$h \times_F g := F(h_{(1)}, g_{(1)}) (h_{(2)} g_{(2)}) F^{-1}(h_{(3)}, g_{(3)})$$

- ▶ Simultaneously deforms all H -covariant constructions as functorial isomorphism of categories of left comodules:

$$\mathcal{Q}_F : {}^H\mathcal{M} \longrightarrow {}^{H_F}\mathcal{M}$$

Notation: $\Delta_L : A \longrightarrow H \otimes A$ left coaction of H on A ,

$$\Delta_L(a) := a^{(-1)} \otimes a^{(0)}$$

Comodule twisting of algebras

- ▶ Trivial “flip” braiding on monoidal category ${}^H\mathcal{M}$:

$$\Psi : A \otimes B \longrightarrow B \otimes A, \quad \Psi(a \otimes b) = b \otimes a$$

- ▶ Twist into new braiding on ${}^{H_F}\mathcal{M}$:

$$\Psi_F : A_F \otimes B_F \longrightarrow B_F \otimes A_F, \quad \Psi_F(a \otimes b) = F^{-2}(b^{(-1)}, a^{(-1)}) (b^{(0)} \otimes a^{(0)})$$

- ▶ A — H -comodule algebra $\implies A_F = Q_F(A)$ — H_F -comodule algebra with new product:

$$a \cdot b := F(a^{(-1)}, b^{(-1)}) (a^{(0)} b^{(0)})$$

- ▶ A, B algebras in ${}^{H_F}\mathcal{M} \implies$ so is braided tensor product $A \underline{\otimes} B$ with:

$$(a \otimes b) \cdot (c \otimes d) = a \Psi_F(b \otimes c) d$$

Noncommutative algebraic torus $T_\theta = (\mathbb{C}_\theta^\times)^n$

- ▶ $H := \mathbb{C}(t_1, \dots, t_n) = A(T)$ generated by $t^p := t_1^{p_1} \cdots t_n^{p_n}$, $p \in \mathbb{Z}^n$ with:

$$\Delta(t^p) = t^p \otimes t^p, \quad \epsilon(t^p) = 1, \quad S(t^p) = t^{-p}$$

- ▶ **Cocycle:** $F(t_i, t_j) = \exp\left(\frac{i}{2} \theta_{ij}\right) =: q_{ij}$, $\theta_{ij} = -\theta_{ji} \in \mathbb{C}$
 $H = H_F$ as Hopf algebras, but category of H -comodules twisted
- ▶ $\Delta : H \longrightarrow H \otimes H$ makes H into comodule algebra in ${}^H\mathcal{M}$, so cotwisted torus has:

$$t_i \cdot t_j = F(t_i, t_j) t_i t_j = F^2(t_i, t_j) t_j \cdot t_i = q_{ij}^2 t_j \cdot t_i$$

Noncommutative torus $A(T_\theta)$ as object of ${}^H\mathcal{M}$

Quantization of toric varieties $X \longrightarrow X_\theta$

(Ingalls)

- ▶ Noncommutative affine toric varieties $\sigma \longmapsto A(U_\theta[\sigma])$
finitely-generated H_F -comodule subalgebras of $A(T_\theta)$
- ▶ **Example:** $A(\mathbb{C}_\theta^n) = \mathbb{C}_\theta[t_1, \dots, t_n]$, $t_i t_j = q_{ij}^2 t_j t_i$
“Algebraic Moyal plane”:

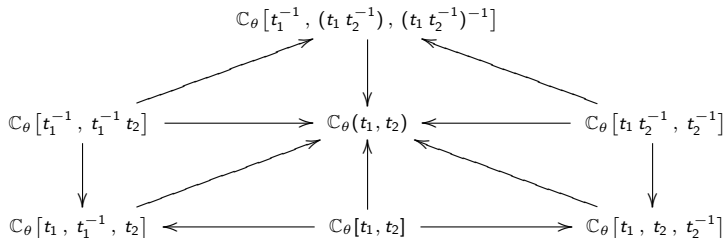
$$t_i \longmapsto z_i = \log t_i, \quad [z_i, z_j] = i\theta_{ij}$$

Generally modulo relations

- ▶ Gluing rules \implies algebra automorphisms in category ${}^{H_F}\mathcal{M}$
- ▶ Uses **same fan**, deforms coordinate algebra of each cone σ

Noncommutative projective plane \mathbb{P}_θ^2

- ▶ Maximal cones: $U_\theta[\sigma_i] \cong \mathbb{C}_\theta^2$, $i = 1, 2, 3$
- ▶ Edges: $U_\theta[\sigma_i \cap \sigma_{i+1}] \cong$ noncommutative projective line \mathbb{P}_θ^1 :
 $w_1 w_2 = q^2 w_2 w_1$, $w_1 w_2^{-1} = q^{-2} w_2^{-1} w_1$, $q := q_{12}$
- ▶ Gluing morphisms:



Homogeneous coordinate algebras

(Auroux, Katzarkov & Orlov '08)

$$w_1 w_2 = q^2 w_2 w_1, \quad w_1 w_3 = w_3 w_1, \quad w_2 w_3 = w_3 w_2$$

- ▶ $A = \mathbb{C}_\theta[w_1, w_2, w_3]$ **graded** algebra object in ${}^{H_F}\mathcal{M}$
- ▶ Degree 0 left Ore localization $A[w_i^{-1}]_0 \cong A(U_\theta[\sigma_i])$
- ▶ $A \longrightarrow A_{\ell_\theta} := A/A \cdot w_3$ noncommutative line $\ell_\theta \cong \mathbb{P}_\theta^1 \hookrightarrow \mathbb{P}_\theta^2$
“at infinity” (given by $w_3 = 0$)
- ▶ Finitely-generated graded right A -modules M
 \iff “coherent sheaves” on \mathbb{P}_θ^2
Projective \iff “bundles”
Torsion-free (no finite-dim. submodules) \iff embeds in a bundle

Instanton moduli spaces $M_\theta(r, k)$

- ▶ **Framed modules:** A -module $M \implies A_{\ell_\theta}$ -module
 $M_{\ell_\theta} = M/M \cdot w_3$; trivialize $M_{\ell_\theta} =$ free A_{ℓ_θ} -module
- ▶ $M_\theta(r, k) =$ isomorphism classes of framed torsion-free A -modules with fixed trivialization $M_{\ell_\theta} \cong A_{\ell_\theta}^{\oplus r}$, $\dim_{\mathbb{C}} \text{Ext}^1(A, M(-1)) = k$

- ▶ **Invariants:**

$\text{rank}(M) =$ max. no. non-zero direct summands of $M = r$

$$\chi(M) = \sum_{p \geq 0} (-1)^p \dim_{\mathbb{C}} \text{Ext}^p(A, M) = r - k$$

Euler characteristic

Noncommutative ADHM construction

- ▶ Matrices $B_1, B_2 \in \text{Mat}_{k \times k}(\mathbb{C})$, $I \in \text{Mat}_{k \times r}(\mathbb{C})$, $J \in \text{Mat}_{r \times k}(\mathbb{C})$ satisfying:

1. Noncommutative complex ADHM equation:

$$[B_1, B_2]_\theta + IJ = 0$$

$$[B_1, B_2]_\theta := B_1 B_2 - q^{-2} B_2 B_1 \quad \text{braided commutator}$$

2. Stability: $V \subsetneq \mathbb{C}^k$, $B_i(V) \subset V$, $\text{im}(I) \subset V \implies V = 0$

- ▶ **Theorem:** $M_\theta(r, k) = \left\{ B_1, B_2, I, J \right\}$ modulo free proper action of $GL(k, \mathbb{C})$:

$$B_i \mapsto g B_i g^{-1}, \quad I \mapsto g I, \quad J \mapsto J g^{-1}; \quad g \in GL(k, \mathbb{C})$$

Noncommutative Klein quadric $\text{Gr}_\theta(2; 4) \hookrightarrow \mathbb{P}_\Theta^5$

(Lauve '06)

- ▶ Braided exterior algebra $\bigwedge_\theta^2 \mathbb{C}^4$ in ${}^{H_F} \mathcal{M}$ given by cocycle twist
- ▶ Spanned by minors Λ^J , $J = (j_1 j_2)$, $1 \leq j_1, j_2 \leq 4$:

$$\Lambda^J \Lambda^K = q_{j_1 k_1}^2 q_{j_1 k_2}^2 q_{j_2 k_1}^2 q_{j_2 k_2}^2 \Lambda^K \Lambda^J$$

- ▶ Existence of embedding $\text{Gr}_\theta(2; 4) \hookrightarrow \mathbb{P}_\Theta^5 \cong \mathbb{P}(\bigwedge_\theta^2 \mathbb{C}^4)$:

$$\Theta^{JK} = \theta^{j_1 k_1} + \theta^{j_1 k_2} + \theta^{j_2 k_1} + \theta^{j_2 k_2}$$

- ▶ Noncommutative Plücker relation:

$$\Lambda^{(12)} \Lambda^{(34)} - q_{23}^2 \Lambda^{(13)} \Lambda^{(24)} + q_{24}^2 q_{34}^2 \Lambda^{(14)} \Lambda^{(23)} = 0$$

Noncommutative sphere S_θ^4

- ▶ $A(S_\theta^4) := \mathbb{R}$ -algebra generated by $*$ -involution on $A(\text{Gr}_\theta(2; 4))$:

$$\Lambda^{(13)\dagger} = q\Lambda^{(24)}, \Lambda^{(14)\dagger} = -q^{-1}\Lambda^{(23)}, \Lambda^{(12)\dagger} = \Lambda^{(12)}, \Lambda^{(34)\dagger} = \Lambda^{(34)}$$

with $q_{12} = q_{21}^{-1} =: q \in \mathbb{R}$, $q_{ij} = 1$ otherwise

- ▶ Open affine subvariety $\mathbb{R}_\theta^4 \subset S_\theta^4$ given by degree 0 right Ore localization $A(\text{Gr}_\theta(2; 4))[\Lambda^{(34)^{-1}}]_0 \cong \mathbb{C}[\xi_1, \bar{\xi}_1, \xi_2, \bar{\xi}_2]$ with:

$$\xi_1 \bar{\xi}_1 = q^2 \bar{\xi}_1 \xi_1, \quad \xi_2 \bar{\xi}_2 = q^{-2} \bar{\xi}_2 \xi_2$$

$$\xi_1 \xi_2 = q^2 \xi_2 \xi_1, \quad \bar{\xi}_1 \bar{\xi}_2 = q^{-2} \bar{\xi}_2 \bar{\xi}_1$$

$$\xi_1 \bar{\xi}_2 = \bar{\xi}_2 \xi_1, \quad \xi_2 \bar{\xi}_1 = \bar{\xi}_1 \xi_2$$

$$\xi_1^\dagger = q^{-1} \bar{\xi}_1, \quad \xi_2^\dagger = -q^{-1} \bar{\xi}_2$$

Noncommutative twistor transform

- ▶ Twistor correspondence:

$$\begin{array}{ccc}
 & A(\mathbb{F}l_\theta(1, 2; 4)) & \\
 \nearrow^{p_1} & & \nwarrow^{p_2} \\
 A(\mathbb{P}_\theta^3) & & A(\text{Gr}_\theta(2; 4))
 \end{array}$$

- ▶ $A(\mathbb{P}_\theta^3)$ = “noncommutative twistor algebra”;
 $\mathbb{F}l_\theta(1, 2; 4)$ = noncommutative partial flag variety:

$$A(\mathbb{P}_\theta^3) \otimes A(\text{Gr}_\theta(2; 4)) \longrightarrow A(\mathbb{F}l_\theta(1, 2; 4))$$

- ▶ Construct instantons on S_θ^4 using **twistor transform**:

$$\left\{ A(\mathbb{P}_\theta^3)\text{-modules} \right\} \longrightarrow \left\{ A(\text{Gr}_\theta(2; 4))\text{-modules} \right\}$$

$$M \longmapsto p_2^* p_{1*}(M) \quad , \quad p_{1*}(M) = \left[M \otimes_{A(\mathbb{P}_\theta^3)} A(\mathbb{F}l_\theta(1, 2; 4)) \right]_{\text{diag}}$$

Self-conjugate instanton modules

- ▶ Quaternion structure on $A(\mathbb{P}_\theta^3)$:

$\mathcal{J}(w_1, w_2, w_3, w_4) = (w_2, -w_1, w_4, -w_3)$ induces
 $M \longmapsto M^\dagger := \mathcal{J}^\bullet(M)^\vee$ on $A(\mathbb{P}_\theta^3)$ -modules

- ▶ On ADHM data, $(B_1, B_2, I, J) \longmapsto (-B_2^\dagger, B_1^\dagger, -J^\dagger, I^\dagger)$, obey
noncommutative real ADHM equation:

$$[B_1, B_1^\dagger]_\theta + q^{-2} [B_2, B_2^\dagger]_{-\theta} + I I^\dagger - J^\dagger J = 0$$

- ▶ **Theorem:** $\left\{ \begin{array}{l} \text{torsion-free } M \cong M^\dagger, \quad M_{\ell_\theta} \cong A_{\ell_\theta}^{\oplus r}, \\ \text{Ext}^1(A(\mathbb{P}_\theta^3), M(-2)) = 0 \end{array} \right\} \iff \left\{ \text{real ADHM data} \right\} / U(k)$

- ▶ Restriction to $\mathbb{P}_\theta^2 \implies$ anti-self-dual connections on a canonical
 “instanton bundle”

Construction of noncommutative instantons

- ▶ Twistor transform of self-conjugate $A(\mathbb{P}_\theta^3)$ -module M gives module over $A(\text{Gr}_\theta(2; 4))$. Restricting to \mathbb{R}_θ^4 gives right $A(\mathbb{R}_\theta^4)$ -module:

$$\mathcal{N} = \ker \mathcal{D}, \quad \mathcal{D} = \begin{pmatrix} B_1 - q^{-1} \xi_1 & B_2 - q \xi_2 & I \\ -B_2^\dagger - q^{-1} \bar{\xi}_2 & B_1^\dagger - q \bar{\xi}_1 & -J^\dagger \end{pmatrix}$$

- ▶ By stability:
 - ▶ \mathcal{D} = surjective morphism of free $A(\mathbb{R}_\theta^4)$ -modules
 - ▶ $\Delta = \mathcal{D} \mathcal{D}^\dagger =$ isomorphism
 - ▶ \mathcal{N} = finitely-generated and projective of rank r with projection $P = 1 - \mathcal{D}^\dagger \Delta^{-1} \mathcal{D}$, $P^2 = P = P^\dagger$
- ▶ Using canonically defined differential structure $\Omega^\bullet(\mathbb{R}_\theta^4)$ gives **instanton connection**:

$$\nabla := P \circ d$$

with curvature $F_\nabla = \nabla^2 = P (dP)^2$