Geometric modular action and diffeomorphism invariance

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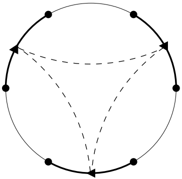


Subtitle:

How spacetime symmetries are encoded
in local von Neumann algebras

Outline

Joint work (Rev. Math. Phys. **22** (2010) 331-354) with **Roberto Longo** (Rome) and **Pierre Martinetti** (Göttingen/Rome)

1. Geometric modular action
2. Chiral CFT and diffeomorphisms
3. **Modular mixing** in free Fermi theory
4. **Charge splitting**
5. 2D Boundary CFT
6. Conclusions

References

References

- Y. Kawahigashi, R. Longo: Noncommutative spectral invariants and black hole entropy, Commun. Math. Phys. **257** (2005) 193–225
- H. Casini, M. Huerta: Reduced density matrix and internal dynamics for multicomponent regions, Class. Quant. Grav. **26** (2009) 185005
- R. Longo, KHR: Local fields in boundary conformal QFT, Rev. Math. Phys. **16** (2004) 909–960

GEOMETRIC MODULAR ACTION

Geometric modular action

Modular theory *[Tomita-Takesaki]*

For faithful normal states φ on von Neumann algebras M: The closable operators $S\, m\Omega_\varphi := m^*\Omega_\varphi$ define assignment

$$(M, \varphi) \mapsto S_{M,\varphi} = J_{M,\varphi} \cdot \Delta^{\frac{1}{2}}_{M,\varphi}.$$

$$\sigma_t := \mathrm{Ad}\,\Delta^{it}, \qquad j := \mathrm{Ad}\,J.$$

Theorem: $\sigma_t(M) = M, \qquad j(M) = M'.$

Local structure of AQFT: Let $M \equiv A(O)$. Then

$$Q \subset O \quad \Rightarrow \quad A(Q) \subset M.$$

Geometric modular action:

$$\sigma_t(A(Q)) \stackrel{!}{=} A(f_t Q).$$

Geometric modular action (ct'd)

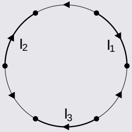
Physical relevance of MT and GMA

- Thermal equilibrium [Haag-Hugenholtz-Winnink]
- Unruh effect, Hawking radiation [Unruh, Sewell]
- Modular localization [Schroer]
- **Reconstruction of symmetries and spacetime** [Kähler-Wiesbrock, Buchholz-Summers, . . .]
- Quantum information [Casini-Huerta]

Geometric modular action (ct'd)

Geometric modular action has been established for:

- wedge algebras in the vacuum state [Bisognano-Wichmann]
- lightcone algebras and doublecone algebras in the conformal vacuum state [Buchholz,Hislop-Longo]
- chiral CFT multi-interval algebras in specially manufactured states [Kawahigashi-Longo, 2005]



The modular action is "fuzzy" for:

- massive doublecone algebras in the vacuum state [Figliolini-Guido,Saffary,Casini-Huerta]
- conformal doublecone algebras in KMS thermal states [Borchers-Yngvason]

Geometric modular action (ct'd)

How can the "fuzzyness" be described in the general case?
Is the flow described by a pseudo-differential operator [Schroer-Wiesbrock]?

Multi-region algebras and boundary CFT exhibit a different kind of fuzzyness in the vacuum state:

- Mixing [Casini-Huerta]
- Charge splitting [LMR] (see below)

Why multi-regions?

- Quantum information (entanglement)
- Boundary CFT
- Modular origin of diffeomorphisms

CHIRAL CFT AND DIFFEOMORPHISMS

Chiral CFT

Setting

- Chiral fields: $\phi(t,x) = \phi(t+x)$ or $\phi(t-x)$
- Local algebras $I \mapsto A(I)$, $I \subset \mathbb{R}$
- Cayley transformation $\mathbb{R} \to S^1$
- Local algebras $I \mapsto A(I)$, $I \subset S^1$
- Diffeomorphism invariance by automorphisms
- Generators: Virasoro algebra ($c > 0$)
- Vacuum state breaks diffeomorphism invariance
- ...preserves Möbius invariance (L_{-1}, L_0, L_{+1})
- Assume complete rationality (Kawahigashi-Longo-Müger)

Modular Theory and chiral CFT

Theorem: *(Guido-Longo 1996)*

For an interval $I \subset S^1$, the modular group of $(A(I), \Omega)$ equals $\Delta^{it} = U(\Lambda_I(-2\pi t))$, where $\Lambda_I \subset$ Möb is the one-parameter group preserving I ($=$ dilations, if $I = \mathbb{R}_+$).

Hence the modular groups of any three intervals generate the Möbius group.
For two adjacent intervals $I_1 \cup I_2 = I \setminus \{\text{point}\}$, $(A(I_1) \subset A(I), \Omega)$ is a standard co-normal **half-sided modular inclusion**.

Theorem: *(Guido-Longo-Wiesbrock 1998)*

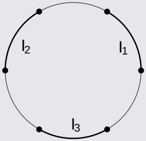
For any standard co-normal half-sided modular inclusion $(N \subset M, \Phi)$, the modular groups of M, N and $N' \cap M$ generate a positive-energy representation of Möb, and hence a chiral CFT on S^1 with Φ as the vacuum vector and $M = A(I)$, $N = A(I_1)$.

Diffeomorphism invariance in chiral CFT

Modular origin of Diff(S^1)?

- n-Möbius groups generated by $\frac{1}{n}L_{-n}, \frac{1}{n}L_0 + \frac{c}{24}\frac{n^2-1}{n}, \frac{1}{n}L_{+n}$.
- subgroups $\subset$ Diff(S^1), together generate Diff(S^1).
- cannot be modular groups in the vacuum state, because ω is not invariant:
- $\gamma(T(x)) = \gamma'(x)^2 \cdot T(\gamma(x)) - \frac{c}{24\pi} D\gamma(x) \cdot \mathbf{1}$, while $\omega(T(x)) = 0$.

Idea (Schroer):



- Möb$^{(n)}$ contain $\Lambda^{(n)}(t)(z) = \sqrt[n]{\Lambda_I(t)(z^n)}$ for "symmetric" n-intervals $E = \sqrt[n]{I}$.
- Could be modular groups of n-intervals $E = I_1 \dot\cup \ldots \dot\cup I_n$ in suitable non-vacuum states.

Modular origin of Möb$^{(n)}$

KL product states

Split isomorphism:

$$\chi : A(E) = \bigvee_{k=1}^{n} A(I_k) \to \bigotimes_{k=1}^{n} A(I_k)$$

Product states on $A(E)$:

$$\varphi_E := \Big(\bigotimes_{k=1}^{n} \varphi_k \Big) \circ \chi$$

Kawahigashi-Longo product states on symmetric n-intervals:

$$\varphi_k = \omega \circ \mathrm{Ad}\, U(h_k), \qquad h_k(z) = z^n \quad \text{for} \quad z \in I_k$$

$\omega =$ vacuum state, $U(h)$ implement diffeomorphisms h.
Extension to $\widehat{A}(E) = A(E')'$ by expectation $\varepsilon_E : \widehat{A}(E) \to A(E)$

$$\widehat{\varphi}_E := \varphi_E \circ \varepsilon_E$$

Multi-interval algebras in chiral CFT (ct'd)

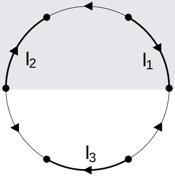
Theorem: *(Kawahigashi-Longo 2005)*

The modular automorphism groups $\sigma_t^{\varphi_E}$ and $\sigma_t^{\widehat{\varphi}_E}$ are implemented on $A(E)$ resp. $\widehat{A}(E)$ by the unitary one-parameter group $V(t)$ of the n-dilatations in Diff(S^1)

$$f_t(z) = \sqrt[n]{\Lambda_I(-2\pi t)(z^n)}.$$

Hence, we have geometric modular action on E. Moreover the n-Möbius generators $D_I^{(n)}$ of $V_I(t)$ are of modular origin:

$$-2\pi D_I^{(n)} = \log\Big(\frac{d\widehat{\varphi}_E}{d\varphi_{E'}}\Big) + \frac{n-1}{2}\log\mu_A.$$



E.g., $D_I^{(n)} = \frac{i}{n}(L_n - L_{-n})$ for $I = S^1_+$.

MODULAR MIXING IN FREE FERMI THEORY

Chiral free Fermi theory = CAR

$$\{\psi(x), \psi(y)\} = \delta(x-y).$$

Vacuum = quasifree state with 2-point function

$$C_2(x,y) = \langle \psi(x)\psi(y) \rangle = \frac{1}{2\pi i(x-y-i\varepsilon)}.$$

[Eckmann-Osterwalder]: Modular operator $\Delta = e^{-K}$ is "2nd quantization" of a one-particle operator.

- This is true for the algebra of **any** region.
- [Figliolini-Guido] have given an abstract characterization of this operator as a resolvent.
- For **interval algebras** $A(I)$: Bisognano-Wichmann property, i.e., $\Delta_I = e^{-2\pi D_I}$, e.g. $D_I = i(L_1 - L_{-1})$ for $I = \mathbb{R}_+ = S^1_+$.

Multi-regions in free Fermi theory

For unions of doublecones in two dimensions

[Casini-Huerta] have computed the one-particle modular Hamiltonian K for free (massive or massless) Fermi fields, by formally solving the resolvent equation

$$C_2\big|_{E\times E} = (1+e^{-K})^{-1}.$$

and derived the Heisenberg differential equation for the modular flow.

- For $m > 0$, the vacuum flow is fuzzy, as expected.
- For $m = 0$, the vacuum flow factorizes into flows on **unions of chiral intervals**, as expected.

Modular mixing (ct'd)

For **multi-interval algebras** $A(E) = \bigvee_k A(I_k)$

Theorem: *(Casini-Huerta 2009)*

As a kernel on $E \times E$:

$$K = -2\pi i \, \frac{\delta(\zeta(x) - \zeta(y))}{x - y}.$$

where for $I_k = (a_k, b_k) \subset \mathbb{R}$, $\zeta : \mathbb{R} \to E$ is the $1 : n$ monotonous map $\mathbb{R} \ni \zeta \to x_k \in I_k$ by

$$e^{\zeta(x)} = -\prod_{k=1}^{n} \frac{x - a_k}{x - b_k}.$$

Modular mixing (ct'd)

Solving the Heisenberg equation

$$\partial_t \, \sigma_t(\psi(x)) = i[K, \sigma_t(\psi(x))],$$

one has to solve $\zeta(x) = \zeta(y)$ with x, y in the same or in different intervals. Accordingly, K has a local term involving $\delta'(x_k - y_k)$, and a nonlocal term involving $\delta(x_k - y_l)$ $(k \neq l)$.

The local term generates a flow $x \to x(t)$, induced by

$$\zeta(t) = \zeta - 2\pi t,$$

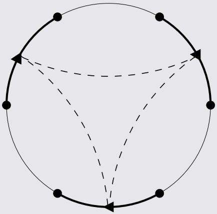
and the nonlocal term is responsible for the **mixing** O_{jk}:

$$\sigma_t(\widehat{\psi}(x_j)) = \sum\nolimits_k O_{jk}(t) \cdot \widehat{\psi}(x_k(t)), \qquad \widehat{\psi}(x) \equiv \sqrt{\frac{dx}{d\zeta}} \cdot \psi(x).$$

Modular mixing (ct'd)

Under the modular evolution, the field in one interval mixes with the field in the other intervals on **associated orbits** $x_k(t)$ in each interval.

$$\sigma_t(\widehat{\psi}(x_j)) = \sum_k O_{jk}(t) \cdot \widehat{\psi}(x_k(t))$$



"**Modular teleportation**"

Modular mixing (ct'd)

Remarks:

- The mixing on top of a GMA is necessary to avoid a contradiction with **Takesaki's Theorem**: invariance of cyclic subalgebras $A(I_k) \subset A(E)$ would imply $A(I_k) = A(E)$.
- For symmetric intervals, the geometric part of the flow is the same as for the Kawahigashi-Longo product states, i.e., the subgroup of $\text{Möb}^{(n)} \subset \text{Diff}(S^1)$ fixing the intervals.
- Therefore, the mixing is the **Connes cocycle** (difference of modular groups for different states on the same algebra).
- The vacuum state is invariant under the modular group, but **not** under $\text{Möb}^{(n)} \subset \text{Diff}(S^1)$. The mixing component therefore restores diffeomorphism invariance via a semidirect product $(\mathbb{R} \subset \text{Diff}(E)) \ltimes (SO(2) \subset SO(n))$.

Modular mixing (ct'd)

Proof by verification of KMS property: *(LMR 2010)*

For symmetric n-intervals the mixing matrix is

$$O(t) = \exp\Big(\big(\xi(t) - \xi(0)\big) \cdot \Omega \Big)$$

($x = \tan\frac{\xi}{2}$, $\Omega =$ universal constant skew symmetric matrix.)

Interesting feature:

$$\Big\langle \sum_k O_{ik}(t) \cdot \widehat{\psi}(x_k(t)) \cdot \sum_l O_{jl}(s) \cdot \widehat{\psi}(y_l(s)) \Big\rangle \sim$$
$$\sim \big\langle \psi(X(t)) \cdot \psi(Y(s)) \big\rangle$$

where $X = e^{\zeta(x_i)}$, $Y = e^{\zeta(y_j)}$ and t and s are the modular parameters on $A(E)$ and on $A(\mathbb{R}_+)$, respectively;
i.e., the mixing matrices O "undo" the $1:n$ map $\mathbb{R}_+ \to E = \bigcup I_k$, $X \mapsto x_k$, and reduces the KMS condition on $E = \sqrt[n]{I}$ to the KMS condition on $\mathbb{R}_+ = \mu(I)$ ($\mu \in$ Möb).

Modular mixing (ct'd)

Speculations

- Geometric modular action on the complement
- Modular conjugation J_E ?
- $(\mathbb{R} \subset \mathrm{Diff}(E)) \ltimes (SO(2) \subset SO(n)) \subset \mathrm{Diff} \ltimes G$?
- Similar in 2 dimensions [CH]. How about 4D ?
- Quantum structure of spacetime ("modular wormholes") ?

CHARGE SPLITTING

Charge splitting

How general is this result?

"Almost all CFTs are sub-CFTs of a free Fermi theory ;-)", e.g., the $c = \frac{1}{2}$ stress-energy tensor ($\mathbb{Z}_2$ fixpoints of a real Fermi field) or the free Bose field ($U(1)$ fixpoints of a complex = two real Fermi fields)

$$A(I) \subset F(I).$$

Modular group for $A(E) \subset F(E)$?

Cannot result by restriction of that for $F(E)$, because the latter does not preserve $A(E)$.

Modular group for $\widehat{A}(E)$?

Cannot result by restriction of that for $F(E)$, because $\widehat{A}(E) \not\subset F(E)$.

Charge splitting

Instead: Being fixpoint subalgebras, there is a vacuum-preserving global conditional expectation

$$\varepsilon : F(I) \to A(I).$$

So, the modular group for $F(I)$ restricts to that for $A(I)$.

But for multi-intervals

$$\begin{array}{ccccc} & & F(E) & & \\ & & \varepsilon \downarrow & & \\ A(E) & \subset & C(E) & \subset & \widehat{A}(E). \end{array}$$

Thus, the modular group restricts to that for $C(E)$.

Charge splitting (ct'd)

The algebra $C(E) = \varepsilon(F(E))$ is **strictly intermediate**: It contains even (neutral) products of Fermi fields in different components of E ("charge transporters" in DHR terminology), but no charge transporters for charges not implemented by the Fermi fields (eg, $h = \frac{1}{16}$).

The modular group of $F(E)$ restricts to $C(E)$. Acting on $A(E) \subset C(E)$, it takes elements of $A(E)$ (neutral in every component I_k) are transformed into elements of $C(E)$ (neutral with charges distributed over the components).

In general, GMA can at best be expected for an intermediate "charge-split" algebra between $A(E)$ and $\widehat{A}(E)$.

- Such an intermediate algebra does not always exist.
- If it exists: How can it be characterized?

Towards a general theory

Proposition: *(LMR 2010)*

If the vacuum modular group of $\widehat{A}(E)$ preserves $A(E)$, then $A(E) = \widehat{A}(E)$.
If the adjoint action of the modular unitaries of $A(E)$ preserves $\widehat{A}(E)$, then $A(E) = \widehat{A}(E)$.

Recall that the difference between $A(E)$ and $\widehat{A}(E)$ signals the presence of nontrivial superselection charges (μ-index, [Kawahigashi-Longo-Müger].
The charge structure therefore poses an obstruction against a simple mixing behaviour as for the free Fermi field.

APPLICATION TO BOUNDARY CFT

Boundary CFT

Two intervals = one doublecone:

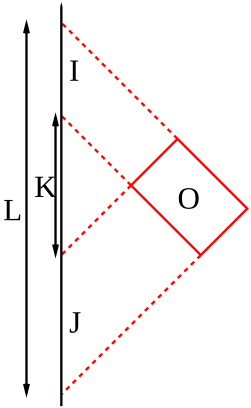
Assume a given chiral CFT. Let

$$O = I \times J \equiv \{(t,x) : t+x \in I, t-x \in J\} \subset M_+.$$

Define

$$A_+(O) := A(K)' \cap A(L).$$

Then $O \mapsto A_+(O)$ is local (even Haag-dual) and contains $A(I) \vee A(J)$.



Generalization: $I \mapsto B(I) \supset A(I)$ relatively local:

$$B_+(O) := B(K)' \cap B(L).$$

- Solves all "Cardy constraints" (local, covariant, OPE).
- = reconstruction of bulk fields from special symmetric FrobA
- = 2D version of AdS-CFT

BCFT (ct'd)

Theorem: *(Longo-KHR 2004)*

Every BCFT is intermediate between $A(I) \vee A(J)$ and $B_+(O) = B(K)' \cap B(L)$ for some chiral net $I \mapsto B(I)$.

- Let $\mathcal{A} = A(I)$, $\mathcal{B} = B(I)$. Then $\mathcal{B}$ is an $\mathcal{A}$-$\mathcal{B}$-module (= subfactor).
- $\mathrm{Rep}(A) \subset \mathrm{End}(\mathcal{A})$ acts on the $\mathcal{A}$-$\mathcal{B}$-modules ($\leftrightarrow$ defects?).
- The orbit of ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}$ is a **nimrep**.
- ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{A}} \simeq \bigoplus_i n_i \, \mathcal{H}_i$ is the Hilbert space of the boundary net B.
- $B_{BCFT}(O)$ as a bimodule of $A(I) \otimes A(J)$ determines a coupling matrix Z.
- $\bigoplus Z_{ij} \, \mathcal{H}_i \otimes \mathcal{H}_j$ is the Hilbert space of the associated bulk net, when the boundary is removed (Longo-KHR 2009).
- Z is a **modular invariant** iff $B_{BCFT}(O) = B(K)' \cap B(L)$ (Böckenhauer-Evans-Kawahigashi, Longo-KHR).
- Admits classification (A completely rational).

Geometric modular action in BCFT

Back to Geometric Modular Action:

$A_+(O) = A(K)' \cap A(L)$ is nothing but $\widehat{A}(E)$.

The KL-result can be applied with the state $\widehat{\varphi}_E$.

In general: $\exists$ unique global conditional expectation

$$\varepsilon : B(L) \to A(L)$$

that commutes with diffeomorphisms. ε maps $B_+(O) \subset B(L)$ into $\widehat{A}(E) \subset A(L)$.

The KL-result can be applied with the state $\widehat{\varphi}_E \circ \varepsilon$.

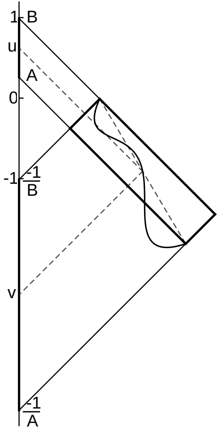
Geometric modular action in BCFT (ct'd)

$\Rightarrow$ geometric modular flow in both intervals $\Rightarrow$:

Resulting modular flow in doublecone $I \times J$

$$\frac{(u-a_1)(u-a_2)}{(u-b_1)(u-b_2)} \cdot \frac{(v-b_1)(v-b_2)}{(v-a_1)(v-a_1)} = \text{const.}$$

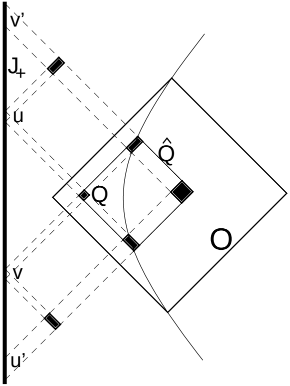
The flow departs from the Möbius flow for doublecones in the vacuum state without boundary. However, the difference is "very small".
(Picture: transverse magnification $\times 100$).



Vacuum GMA in subtheories of free Fermi theories

BCFT with $B_+(O) = C(E) = F(E)^{\mathbb{Z}_2}$

- Local operators $\phi(t, x)$ in the bulk are bi-localized operators $\psi(u)^n\psi(v)^m$ on the boundary ($n + m =$ even).
- The vacuum modular flow of O takes $\psi(u)^n\psi(v)^m$ to $\psi(f_t u)^{n_1}\psi(f_t u')^{n_2}\psi(f_t v)^{m_1}\psi(f_t v')^{m_2}$.
- Rearrange into neutral combinations.
- Hence chiral charge splitting mixes six bulk points in BCFT.



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CONCLUSIONS

Conclusions

Conclusions:

- n-Möbius groups in chiral CFT arise as modular actions in Kawahigashi-Longo product states on n intervals.
- In free Fermi theory, the vacuum state has n-Möbius geometric modular action *plus* **mixing**.
- Mixing restores the invariance of the vacuum state under diffeomorphisms.
- Upon descent to gauge-invariant subtheories, mixing leads to **charge splitting**.
- **The vacuum modular group is sensitive to charge structure.**
- Special application (“two intervals = one doublecone”): **Boundary CFT**

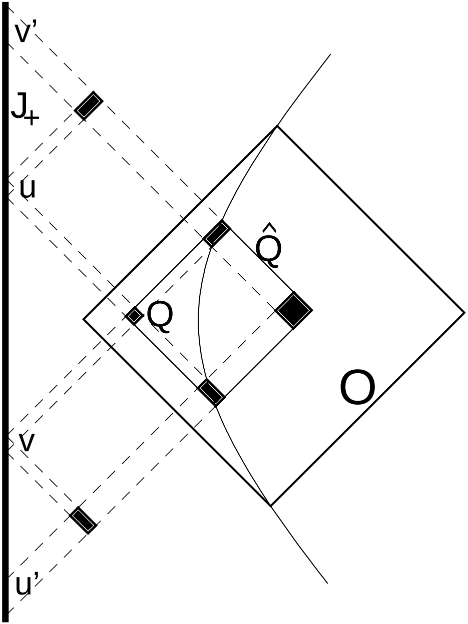
Open questions

Open questions:

- What distinguishes the KL product states among many other states of the same making? (Mixing as a "UV-minimal" Connes cocycle?)
- What is the vacuum modular group for $A(E)$ or $\widehat{A}(E)$?
- How universal are the features of mixing and charge splitting?
- What is the unitary group with positive generator associated with halfsided modular inclusions?
- **What is the symmetry group of the vacuum, generated by modular groups for several multi-intervals?**
- Modular quantum spacetime with modular wormholes?

THANK YOU

BCFT mixing



Back

Note: the algebras of the two regions outside O are actually subalgebras of $B_+(\widehat{Q})$, but they cannot be sharply localized in $\widehat{Q}$ due to the OPE of $\psi(u)\psi(v')$. They belong to the chiral subalgebra of an interval $A(J_+) \subset B_+(\widehat{Q})$.