Sofic equivalence relations and Connes' embedding problem

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Proposition

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Theorem

(Kirchberg '93) Following are equivalent:

- *Connes' Embedding Problem;*
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Proposition

M *embeds in* R^ω *iff* M *embeds in some* $\Pi_{k\to\omega} M_{n_k}(\mathbb{C})$.

Definition

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A group G is hyperlinear iff $\forall\ F \subset G$ finite $\forall\ \varepsilon > 0$, $\exists n\ \exists\Theta : F \to \mathcal{U}(n)$ such that:

- *if* $g, h, gh \in F \quad ||\Theta(g)\Theta(h) - \Theta(gh)||_2 < \varepsilon$;
- *if* $g \in F,\ g \neq e \quad |Tr(\Theta(g))| < 1/2$.

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Notation

- $D_n(\mathbb{C}) \subset M_n(\mathbb{C})$ subalgebra of diagonal matrices;
- $P_n(\mathbb{C}) \subset M_n(\mathbb{C})$ subgroup of permutation matrices.

Question

When $L^\infty(X) \rtimes_\alpha G$ embeds in R^ω?

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Lemma

If $L^\infty(X) \rtimes_\alpha G$ *embeds in* $\Pi_{k\to\omega} M_{n_k}$ *then there exist an embedding such that:* $L^\infty(X) \subset \Pi_{k\to\omega} D_{n_k}$.

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Theorem

(Popa) $\Pi_{k\to\omega} D_{n_k}$ *is a nonseparable MASA in* $\Pi_{k\to\omega} M_{n_k}$.

Proposition

$\mathcal{N}(D_{n_k}) = \mathcal{U}(D_{n_k}) \cdot P_{n_k}$
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Definition

(Normalizer of an algebra) $\mathcal{N}(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$.

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Exemple

Amenable groups, residually finite groups are sofic (free groups).

Definition

A group G is *initially subfinite* if $\forall\ F \subset G$ finite $\exists H$ a finite group and $\exists \Theta : F \to H$ such that if $g, h, gh \in F$ then $\Theta(g)\Theta(h) = \Theta(gh)$.

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Proposition

Class of sofic groups is closed under direct products, amenable extensions, amalgamated products over amenable groups.

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Theorem

(Elek, Lippner) Let G be a sofic group and β a Bernoulli shift action. Then $L^\infty(X) \rtimes_\beta G$ embeds in $\Pi_{k\to\omega} M_{n_k}(\mathbb{C})$.

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Observation

The constructed embedding is such that $L^\infty(X) \subset \Pi_{k\to\omega} D_{n_k}$ and $u_g \in \Pi_{k\to\omega} P_{n_k}$ for any $g \in G$.

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An action α of a group G is called *sofic* if $L^\infty(X) \rtimes_\alpha G$ embeds in $\Pi_{k\to\omega} M_{n_k}$ such that $L^\infty(X) \subset \Pi_{k\to\omega} D_{n_k}$ and $u_g \in \Pi_{k\to\omega} P_{n_k}$ for any $g \in G$.

Proposition

Let α and β be two free orbit equivalent actions (i.e. $L^\infty(X) \rtimes_\alpha G \simeq L^\infty(X) \rtimes_\beta H$). If α sofic then also β sofic.

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For an equivalence relation E let $M(E)$ be the associated von Neumann algebra and $A \subset M(E)$ the diagonal algebra.

Definition

(Elek, Lippner) An equivalence relation E is called *sofic* if there is an embedding of $M(E)$ in some $\Pi_{k\to\omega}M_{n_k}$ such that $A \subset \Pi_{k\to\omega}D_{n_k}$ and $N(A) \subset U(A) \cdot \Pi_{k\to\omega}P_{n_k}$.

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Theorem

(Elek, Lippner) The orbit equivalence relation generated by a Bernoulli shift of a sofic group is sofic.

Theorem

(Elek, Lippner) Treeable equivalence relations are sofic.

Definition

An equivalence relation is treeable iff $\exists\{\varphi_i : A_i \to B_i\}_{i\in\mathbb{N}} \subset [[E]]$ such that:

$$E = \sqcup_{\gamma_{i_1}^{\varepsilon_1}\dots\gamma_{i_n}^{\varepsilon_n}\in\mathbb{F}_\infty} graph\{\varphi_{i_1}^{\varepsilon_1}\dots\varphi_{i_n}^{\varepsilon_n}\}.$$

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Observation

There exists treeable groups (groups for which every free action is treeable). Examples: amenable groups, free groups, $SL_2(\mathbb{Z})$.

Definition

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Proposition

(P) The class $\mathcal{S}$ is closed under amalgamated products over amenable groups.

Exemple

Group $\mathbb{Z} *_{(2,3)\mathbb{Z}} \mathbb{Z} \in \mathcal{S}$ and it is not treeable.