Non-commutative integral forms

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(TB, L El Kaoutit, C Lomp *Non-commutative integral forms and twisted multi-derivations*, JNCG 4:281-312, 2010)

Aim

- To introduce formal theory of integral complexes to noncommutative geometry and formulate explicit Poincaré duality.

Hom-connections

Fix a differential graded algebra ΩA over an algebra A.

Definition

A *hom-connection* is a pair (M, ∇), where M is a right A-module and ∇ is a k-linear map

$$\nabla : \mathrm{Hom}_A(\Omega^1 A, M) \to M,$$

such that, for all $a \in A$, $f \in \mathrm{Hom}_A(\Omega^1 A, M)$,

$$\nabla(f \cdot a) = \nabla(f) \cdot a + f(da),$$

where $f \cdot a \in \mathrm{Hom}_A(\Omega^1 A, M)$ is given by $f \cdot a : \omega \mapsto f(a\omega)$, and $d : \Omega^* A \to \Omega^{*+1} A$ is the differential.

Flat hom-connections

Definition

Define $\nabla_n : \mathrm{Hom}_A(\Omega^{n+1}A, M) \to \mathrm{Hom}_A(\Omega^n A, M)$, by $\nabla_n(f)(\omega) = \nabla(f\cdot\omega) + (-1)^{n+1} f(d\omega)$, where, for all $f \in \mathrm{Hom}_A(\Omega^{n+1}A, M)$, the map $f\cdot\omega \in \mathrm{Hom}_A(\Omega^n A, M)$ is given by $\omega' \mapsto f(\omega\omega')$. The composite $F = \nabla \circ \nabla_1$ is called the *curvature* of (M, ∇).

Lemma

The curvature of a hom-connection is a right A-linear map.

Definition

The hom-connection (M, ∇) is said to be *flat* provided its curvature is equal to zero.

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Integral forms

If ∇ is flat, then $\nabla_n \circ \nabla_{n+1} = 0$ ($\nabla_0 = \nabla$).

Definition

Given a flat hom-connection (A, ∇), the corresponding complex $(\mathfrak{I}^n(A) := \mathrm{Hom}_A(\Omega^{n+1}A, A), \nabla_n)$ is called a *complex of integral forms*.

The cokernel map $\Lambda : A \to \mathrm{coker}\nabla$ is called a ∇*-integral*.

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Classical interpretation

Take a smooth manifold X and a smooth vector bundle E over X. The sections of any vector bundle over X are a (right) module over the algebra of smooth functions $C^\infty(X)$. By the Serre-Swan theorem,

$$\mathrm{Hom}_{C^\infty(X)}(\Gamma(T^*X), \Gamma(E)) \simeq \mathrm{Vect}_X(T^*X, E).$$

$\mathrm{Vect}_X(T^*X, E)$ is a (right) $C^\infty(X)$-module with the fibrewise product ($\mathrm{Vect}_X(T^*X, E)$ can be identified with the module of sections on the hom-bundle $\mathrm{Hom}(T^*X, E)$ over X). With this identification, a hom-connection $(\Gamma(E), \nabla)$ corresponds to a map

$$\nabla : \mathrm{Vect}_X(T^*X, E) \to \Gamma(E),$$

such that, for all $\varphi \in \mathrm{Vect}_X(T^*X, E)$ and $f \in C^\infty(X)$,

$$\nabla(\varphi f) = \nabla(\varphi)f + \varphi \circ df.$$

Manin calls such maps *right connections*.

Classical interpretation

For a smooth compact oriented manifold:

- (canonical) integral forms exist (can be constructed locally);
- the integral coincides with the standard integration over the volume form;
- the complex of integral forms is isomorphic to the de Rham complex (Poincaré duality).

Basic questions

For a non-commutative algebra A, one should ask:

- When do hom-connections exist?
- How to construct hom-connections and integral forms?
- When and if does the Poincaré duality hold?

Existence

Theorem

An A-module M admits a hom-connection with respect to the **universal** *dga if and only if M is injective.*

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Any injective A-module admits a hom-conection with respect to **any** *dga.*

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Twisted multi-derivations

Definition

A *right twisted multi-derivation* in A is a pair (∂, σ), where $\sigma : A \to M_n(A)$ is an algebra homomorphism and $\partial : A \to A^n$ is a k-linear map such that, for all $a \in A$, $b \in B$,

$$\partial(ab) = \partial(a)\sigma(b) + a\partial(b).$$

Free twisted multi-derivations

View $\sigma : A \to M_n(A)$ as an element of $M_n(\mathrm{End}_k(A))$. Write $\bullet$ for the product in $M_n(\mathrm{End}_k(A))$.

Definition

We say that (∂, σ) is *free*, provided there exist algebra maps $\bar\sigma : A \to M_n(A)$ and $\hat\sigma : A \to M_n(A)$ such that

$$\bar\sigma \bullet \sigma^T = \mathbb{I}, \qquad \sigma^T \bullet \bar\sigma = \mathbb{I},$$

$$\hat\sigma \bullet \bar\sigma^T = \mathbb{I}, \qquad \bar\sigma^T \bullet \hat\sigma = \mathbb{I}.$$

Example

σ triangular, with invertible diagonal entries (in $M_n(\mathrm{End}_k(A))$).

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Construction of $\Omega^1 A$

- $\Omega^1 A := A^n_\sigma$,
  a free left A-module $\bigoplus_{i=1}^n A\omega_i$ with basis $\omega_1, \ldots, \omega_n$ and right A-action given by

$$\omega_i a = \sum_j \sigma_{ij}(a)\omega_j, \qquad i = 1, 2, \ldots, n.$$

- $da = \sum_i \partial_i(a)\omega_i$.

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Construction of ∇

Theorem

Let $(\partial, \sigma; \bar{\sigma}, \hat{\sigma})$, be a free right twisted multi-derivation on A, and let $\Omega^1 A$ be the associated first order differential calculus with generators ω_i. Let $\xi_i \in \mathfrak{I}^1(A)$ be given by $\xi_i(\omega_j) = \delta_{ij}$, $i, j = 1, 2, \ldots, n$. Then there exists a unique hom-connection $\nabla : \mathfrak{I}^1(a) \to A$ such that

$$\nabla(\xi_i) = 0,$$

for all $i = 1, 2, \ldots, n$.

Proof. Write $\partial_i^\sigma := \sum_{j,k} \bar{\sigma}_{kj} \circ \partial_j \circ \hat{\sigma}_{ki}$. Then

$$\nabla : \mathrm{Hom}_A(\Omega^1 A, A) \to A, \qquad f \mapsto \sum_i \partial_i^\sigma \left(f(\omega_i) \right). \qquad \square$$

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Example: matrix algebra

Derivation based calculus, the toy model [Dubois-Violette, Kerner, Madore '90]:

- $A = M_n(\mathbb{C})$.
- Take derivations on A and identify them with $sl(n, \mathbb{C})$, by

$$\mathfrak{X}_I(a) = \imath[E_I, a].$$

- Define $\Omega^m A$ as k-multilinear antisymmetric maps

$$sl(n, \mathbb{C})^m \to M_n(\mathbb{C}).$$

- The differential is given by the Koszul formula, in particular

$$da(\mathfrak{X}) = \mathfrak{X}(a).$$

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- $sl(n, \mathbb{C})$ is finite dimensional, hence can identify

$$\mathfrak{I}^1(M_n(\mathbb{C})) = \mathrm{Hom}_A(\Omega^1 A, A) \cong sl(n, \mathbb{C}) \otimes M_n(\mathbb{C}),$$

  and

$$\nabla\left(\sum_l \mathfrak{X}_l \otimes a_l\right) = \sum_l \mathfrak{X}_l(a_l) = \sum_l \imath[E_l, a_l].$$

- ∇ is flat, hence we have a complex of integral forms.
- $\mathrm{Im}(\nabla) = sl(n, \mathbb{C})$, hence $\mathrm{coker}(\nabla) = \mathbb{C}$, and

$$\Lambda(a) = \Lambda\left(\left(a - \frac{1}{n}\mathrm{Tr}(a)\right) + \frac{1}{n}\mathrm{Tr}(a)\right) = \frac{1}{n}\mathrm{Tr}(a).$$

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- Take A to be a Hopf algebra with a bijective antipode.
- Any left covariant differential calculus comes from twisted multiderivations (Woronowicz).
- Any left covariant differential calculus comes from **free** twisted multiderivations. Hence ∇ exists.
- If there is a right integral (Haar measure) h on A, then

Can be extended to quantum principal bundles (principal comodule algebras).

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Example: $SU_q(2)$

In case $A = SU_q(2)$ with 3D calculus:

- ∇ is flat.
- $\Lambda = h$.
- Poincaré duality holds.

The same is true about the quantum Podleś sphere S^2_q (seen as a base of the quantum Hopf fibration).

Other examples include $E_q(2)$ (contraction of $SU_q(2)$), $\mathbb{C}^2_q$.

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