# Line Bundles in noncommutative geometry

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Work in progress - comments welcome!

# Part 1: On line bundles

A (locally trivial) line bundle $L$ on a topological space $X$ is $U \times \mathbb{K}$ ($\mathbb{K}$ being real or complex numbers as appropriate) on an open cover of $X$, with invertible $\mathbb{K}$ valued transition functions.

The dual bundle $L'$ is also a line bundle, and the tensor product bundle $L \otimes L'$ is a trivial bundle.

Endomorphisms of the line bundle are simply given by multiplication by functions on $X$.

Line bundles are classified up to bundle isomorphism by $H^1(X; \mathbb{K}^*)$. If we use a metric on the bundles, we get $H^1(X; S^1)$ for complex line bundles and $H^1(X; \mathbb{Z}/2)$ for real line bundles.

# Finitely generated projective modules

Suppose that $A$ is a possibly noncommutative algebra. A left $A$-module $E$ has dual $E^\circ = {}_A\mathrm{Hom}(E, A)$ (left module maps from $E$ to $A$). Now $E^\circ$ is a right $A$-module by $(\alpha.a)(e) = \alpha(e).a$.

Then $E$ is a finitely generated projective left module of there is a dual basis $e^i \in E$ and $e_i \in E^\circ$ (or $1 \le i \le n$) so that for all $e \in E$,

$$e \;=\; e_i(e).e^i \;.$$

If $E$ is also a right $A$-module then $E^\circ$ has a left $A$-action by $(a.\alpha)(e) = \alpha(e.a)$.

# Evaluations and coevaluations

For an $A$-bimodule $E$ which is finitely generated projective as a left module, there are evaluation and coevaluation maps

$$\mathrm{ev} : E \underset{A}{\otimes} E^\circ \to A \;, \quad \mathrm{coev} : A \to E^\circ \underset{A}{\otimes} E \;.$$

The coevaluation is given in terms of the dual basis by

$$\mathrm{coev}(1_A) \;=\; \sum e_i \otimes e^i \;.$$

These obey the conditions

$$\begin{aligned} (\mathrm{ev} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{coev}) &= \mathrm{id} : E \to E \;, \\ (\mathrm{id} \otimes \mathrm{ev})(\mathrm{coev} \otimes \mathrm{id}) &= \mathrm{id} : E^\circ \to E^\circ \;. \end{aligned}$$

# (Strict) Morita contexts

Morita contexts are designed to give equivalences between categories of (left) modules of algebras. (I.e. functors giving isomorphisms up to natural equivalence.)

A strict Morita context for algebras $A$ and $C$ is an $A-C$ module $N$ and a $C-A$ module $M$ so that $N \otimes_C M \cong A$ as $A$-bimodules, and $M \otimes_A N \cong C$ as $C$-bimodules, and an associativity condition.

The corresponding functor from ${}_A\mathbb{M}$ (left $A$-modules) to ${}_C\mathbb{M}$ is then $E \mapsto M \otimes_A E$, and from ${}_C\mathbb{M}$ to ${}_A\mathbb{M}$ is $F \mapsto N \otimes_C F$.

# Line modules

A good definition of a 'line module' or rank one module is a strict Morita context from $A$ to $A$ - that is a pair of $A$ bimodules $L$ and $L^\circ$ so that $L \otimes_A L^\circ \cong A$ and $L^\circ \otimes_A L \cong A$ with the associativity condition.

By the usual machinery (Bass) this is equivalent to an $A$ bimodule $L$ which is finitely generated projective as a left module and where the evaluation map $\mathrm{ev} : L \otimes_A L^\circ \to A$ and the coevaluation $\mathrm{coev} : A \to L^\circ \otimes_A L$ are isomorphisms.

If we have coevaluation being an isomorphism, then we can consider this to be a 'weak line module'. By the usual machinery, if evaluation is onto, then it is an isomorphism.

**Proposition:** Suppose that $L$ is a weak line module, and that $E$ is a left module. Given $T : L \to L \otimes_A E$ a left module map, there is a $e \in E$ so that $T(x) = x \otimes e$.

# The $\mathbb{N}$ and $\mathbb{Z}$ graded tensor algebras

For any $A$-bimodule $E$, we can define an $\mathbb{N}$ graded tensor algebra by

$$T_{\mathbb{N}}(E) \; = \; A \oplus E \oplus (E \underset{A}{\otimes} E) \oplus (E \underset{A}{\otimes} E \underset{A}{\otimes} E) \oplus \dots \; .$$

The associative product on the algebra $T_{\mathbb{N}}(E)$ is just $\otimes_A$.

We shall also consider a $\mathbb{Z}$ graded object $T_{\mathbb{Z}}(E)$ by

$$T_{\mathbb{Z}}(E)^n \; = \; \begin{cases} A & n = 0 \\ E^{\otimes_A^n} & n > 0 \\ (E^\circ)^{\otimes_A^{-n}} & n < 0 \end{cases} \; .$$

In general this will not form an algebra under $\otimes_A$ as $E \otimes_A E^\circ \neq A$.

# The product on the $\mathbb{Z}$ graded tensor algebra

**Proposition** If $L$ is a left line $A$-bimodule, then $T_{\mathbb{Z}}(L)$ with product described below gives an associative algebra.

The product on $T_{\mathbb{Z}}(E)^n \otimes T_{\mathbb{Z}}(E)^m \to T_{\mathbb{Z}}(E)^{n+m}$ is given by $\otimes_A$ for $n$ and $m$ of the same sign, and for different signs we use (for $n > 0$)

$$\mathrm{ev}^n : L^{\otimes_A^n} \otimes L^{\circ\,\otimes_A^n} \to A\,, \quad \tilde{\mathrm{ev}}^n : L^{\circ\,\otimes_A^n} \otimes L^{\otimes_A^n} \to A\,.$$

These are defined recursively by $\tilde{\mathrm{ev}} = \mathrm{coev}^{-1}$, and

$$\mathrm{ev}^{n+1} \;=\; \mathrm{ev}\,(\mathrm{id} \otimes \mathrm{ev}^n \otimes \mathrm{id})\,, \quad \tilde{\mathrm{ev}}^{n+1} \;=\; \tilde{\mathrm{ev}}\,(\mathrm{id} \otimes \tilde{\mathrm{ev}}^n \otimes \mathrm{id})\,.$$

Any excess of $L$ or $L^\circ$ are filled in by identities.

# Proof of associativity for the $\mathbb{Z}$ graded tensor algebra

We begin with the simplest cases - we should have

$$\mathrm{ev} \otimes \mathrm{id} = \mathrm{id} \otimes \mathrm{coev}^{-1} : L \underset{A}{\otimes} L^\circ \otimes L \to L ,$$
$$\mathrm{coev}^{-1} \otimes \mathrm{id} = \mathrm{id} \otimes \mathrm{ev} : L^\circ \underset{A}{\otimes} L \otimes L^\circ \to L^\circ .$$

This is part of the definition of strict Morita contexts (see the book by Bass - Algebraic $K$ theory). Alternatively, it can be shown from the definition of evaluation and coevaluation maps. Generalising this to a proof of associativity on $T_{\mathbb{Z}}(L)$ is a not too complicated combinatorial induction argument.

# A star structure on the $\mathbb{Z}$ graded algebra

Remember that a Hermitian structure on an $A$ bimodule $E$ is a bimodule isomorphism $G : \overline{E} \to E^\circ$, giving an inner product $\langle , \rangle : E \otimes_A \overline{E} \to A$ obeying $\langle x, \overline{y} \rangle = \langle y, \overline{x} \rangle^*$. We then define a star operation on $T_{\mathbb{Z}}(E)^{\pm 1}$ by

$$E \xrightarrow{\mathrm{bb}} \overline{\overline{E}} \xrightarrow{\overline{G}} \overline{E^\circ} \,, \quad E^\circ \xrightarrow{G^{-1}} \overline{E} \,.$$

This can be extended to all of $T_{\mathbb{Z}}(E)$ by $(x \otimes y)^* = y^* \otimes x^*$.

After some work, this can be shown to make $T_{\mathbb{Z}}(L)$ (for a line module $L$) into a star algebra.

# Star operations and the Thom construction

Given an $\mathbb{R}^n$ bundle on $X$, the Thom construction adds a point at infinity to each fiber to make an $S^n$ bundle. We have taken a line module $L$ and made a star algebra on which the circle group acts. So have we performed the Thom construction? Not really, but to explain why not we need to look at the star operation again.

We had the star operation mapping $L$ to $L^\circ$. This is the case where $L$ is the canonical bundle of a smooth algebraic variety - the local generator $\mathrm{d}z_1 \wedge \ldots \mathrm{d}z_n$ is mapped to $\mathrm{d}\bar{z}_1 \wedge \ldots \mathrm{d}\bar{z}_n$, the generator of the dual.

In the usual Thom construction, the total space is an $\mathbb{R}^1$ bundle on $X$, and star should map a complex valued function on this space to another function on the same space. In terms of modules, it should map $L$ to $L$. We should use the $\mathbb{N}$ graded tensor algebra.

# How to vanish at infinity, part 1

In the $\mathbb{N}$ graded tensor algebra $T_{\mathbb{N}}(L)$, the summand $L^{\otimes_A^n}$ can be thought of as the functions on the line bundle with polynomial growth of order $n$. The problem is that only zero polynomials vanish at infinity, so we need to make another algebra. There is an obvious bigger algebra, the formal power series algebra

$$\prod_{n\in\mathbb{N}} L^{\otimes_A^n} .$$

But how to take a subalgebra of this corresponding to functions vanishing at infinity? Suppose that $L$ has a non-degenerate Hermitian metric $G$ and a star operation, and let $\xi \in L \otimes_A L$ be the image of $1_A$ under

$$A \xrightarrow{\text{coev}} L^\circ \underset{A}{\otimes} L \xrightarrow{G^{-1}\otimes\text{id}} \overline{L} \underset{A}{\otimes} L \xrightarrow{\star^{-1}\otimes\text{id}} L \underset{A}{\otimes} L .$$

Then $\xi$ is central for the $A$ actions and of ‘quadratic growth’.

# How to vanish at infinity, part 2

Now we check that $\xi$ is central in $T_{\mathbb{N}}(L)$, i.e. that $e \otimes \xi = \xi \otimes e \in L^{\otimes_A^3}$ for all $e \in L$. Take $\alpha \in L^\circ$, and consider $T : L \to L \otimes_A L$,

$$T(e) \;=\; \xi.\mathrm{ev}(e \otimes \alpha)\ .$$

This is a left module map, so by a previous result $T(e) = e \otimes f$ for some $f \in L$. Now suppose that the following is the identity:

$$A \xrightarrow{\mathrm{coev}} L^\circ \underset{A}{\otimes} L \xrightarrow{(G\star)^{-1} \otimes G\star} L \underset{A}{\otimes} L^\circ \xrightarrow{\mathrm{coev}} A\ .$$

Now applying $\mathrm{ev}(\mathrm{id} \otimes G\star)$ to $T(e)$ gives

$$\mathrm{ev}(e \otimes \alpha) \;\;=\;\; \mathrm{ev}(e \otimes G \star (f))\ ,$$

and $\mathrm{ev}$ being an isomorphism does the rest.

Note that the assumption on $G\star$ above is reasonable, and that some sort of assumption is expected!

# How to vanish at infinity, part 3

Now inside the formal power series algebra we consider the subalgebra generated by things of the form

$$\frac{x}{(1+\xi)^m}\,, \quad x \in L^{\otimes_A^n} \text{ for } 0 \leq n < 2\,m\,.$$

As $\xi$ is central these can be multiplied and added (by taking the common denominator) quite simply. This is an algebra of functions 'vanishing at infinity' on the line bundle.

But this is not yet really the Thom construction. For that, we would expect that, starting with a $C^*$ algebra, we should get another $C^*$ algebra. There is some more work to do... However it should be expected that a candidate Hilbert space could be constructed from the algebra above. The functions there are of 'degree $\leq -1$ on the line bundle', and so *ought* to be square integrable...

# The Picard group

The tensor product of two line bundles on a (nice) topological space $X$ is another line bundle. The identity for this operation is the trivial bundle, and every line bundle has an inverse (dual).

Just the same works for line modules, where we set $(L \otimes_A M)^\circ = M^\circ \otimes_A L^\circ$.

Line bundles are classified up to bundle isomorphism by $H^1(X; \mathbb{K}^*)$. If we use a metric on the bundles, we get $H^1(X; S^1)$ for complex line bundles and $H^1(X; \mathbb{Z}/2)$ for real line bundles.

In the noncommutative case, does this (isomorphism classes of line modules) give a reasonable definition of $H^1(A; S^1)$ and $H^1(A; \mathbb{Z}/2)$??

# Part 2: On principal bundles

A principal bundle on a (nice) topological space $X$ is a locally trivial bundle $P$ with a group $G$ action transitively and freely on the fibers. (I will stay with $G$ being a finite abelian group, or at worst compact.)

Given a representation $V$ of $G$, we can construct a vector bundle with fiber $V$ associated to the principal bundle.

$H^1(X; G)$ is in 1-1 correspondence with isomorphism classes of principal $G$ bundles on $X$. The easiest way to see this is to use Cech cohomology, where the data for a class in $H^1(X; G)$ is the same as that required to build a principal $G$ bundle.

# Addition of principal bundles

The addition on $H^1(X; G)$ in terms of the bundles can be seen as follows: Take principal $G$ bundles $P$ and $Q$. Now $G$ acts diagonally on the cross product bundle $P \times_X Q$, and on the second factor separately. These actions commute if $G$ is abelian. If we quotient $P \times_X Q$ by the diagonal action, the action on the second factor makes the quotient into a principal bundle, the 'sum' bundle.

The principal bundle $P$ has a $G$ action, so $C(P)$ has a $C(G)$ coaction. In fact $C(P)$ is a Hopf Galois extension of $C(X)$.

# Hopf Galois extensions

Let $B$ be an algebra and $H$ a Hopf algebra. Suppose that there is a right coaction : $B \to B \otimes H$ denoted by $b \mapsto b_{[0]} \otimes b_{[1]}$ making $B$ into a comodule algebra, i.e.

$$b\, b' \longmapsto b_{[0]}\, b'_{[0]} \otimes b_{[1]}\, b'_{[1]} \; .$$

The invariant part of $B$ is an algebra, $A$

$$A \;=\; B^{coH} \;=\; \{b \in B : b_{[0]} \otimes b_{[1]} = b \otimes 1_H\} \; .$$

The canonical map $\mathrm{can} : B \otimes_A B \to B \otimes H$ is defined by $p \otimes q \mapsto p\, q_{[0]} \otimes q_{[1]}$. $B$ is called a Hopf Galois extension of $A$ if the canonical map is a 1-1 correspondence. (We will always assume that $H$ has an integral, so the difficult part is showing that the map is onto.)

# Theorem

There is a 1-1 correspondence between

1) Automorphisms of the category ${}_A\mathbb{M}$ of left $A$ modules
2) Left line modules over $A$
3) Group algebra of $\mathbb{Z}$ Hopf Galois extensions of $A$

**Proof:** Contained in the previous constructions. The (1) $\Leftrightarrow$ (2) parts are standard, and (3) $\Rightarrow$ (2) is noted by Caenepeel. The (2) $\Rightarrow$ (3) part is the $\mathbb{Z}$ graded tensor algebra construction.

Note that line bundles are classified by $H^1$ with coefficients in the circle group, so the circle acts. This means that the dual to the circle - the group algebra of the integers - coacts, and the Hopf Galois condition is phrased in terms of coactions.

The original line bundle can be recovered as an associated bundle to the Hopf Galois extension.

# The $\mathbb{Z}/p$ cohomology of an algebra???

Why is $K$-theory a good thing for $C^*$ algebras? Firstly because it gives a good extension of a useful commutative idea. Secondly because it is calculable in lots of cases!! Thirdly, it remains useful even for very noncommutative algebras.

The first and second properties are linked to the useful idea of sections of vector bundles on $X$ being modules for $C(X)$. If we just has a local $GL(n, \mathbb{C})$ definition for bundles, things would be more difficult. The moral is to look for a good global definition for something which is, *a priori*, local.

**Conjecture??:** For a cocommutative Hopf algebra $H$, *define* $H^1(A; H')$ as isomorphism classes of $H$ Hopf Galois extensions of $A$. (Here $H'$ is the dual Hopf algebra.)

Yes, a definition cannot be a conjecture. So what this means is: (1) Is it an extension of the classical ideas? (2) Is it calculable? (3) Is it useful?

# Mapping properties of Hopf Galois extensions

If we have a map of groups, then there is a corresponding map of $H^1(X, -)$. There is a corresponding construction for Hopf Galois extensions, given a Hopf algebra map. We suppose that all our Hopf algebras have integrals.

Suppose that $\theta : K \to H$ is a Hopf algebra map. Then for $B$ a $H$ Hopf Galois extension of $A$, define $\theta^* B$ to be $B \square^H K$. Here

$$B \square^H K = \left\{ b \otimes k \in B \otimes K \middle| b_{[0]} \otimes b_{[1]} \otimes k = b \otimes \theta(k_{(1)}) \otimes k_{(2)} \right\}$$

and the coaction on $B \square^H K$ is coproduct on the last factor. This makes $B \square^H K$ into a $K$ Hopf Galois extension of $A$.

# The coefficient long exact sequence??

Given a short exact sequence of groups, there is a long exact sequence of $H^n(X, -)$. To decide what this means in terms of algebras, we first have to say what the 'zero principal bundle' is - this is the trivial extension $A \otimes H$. The 'zero' Hopf algebra is the field $\mathbb{K}$, and the 'zero map' from $H$ to $\mathbb{K}$ is $h \mapsto \epsilon(h).1$. Now if we have maps of Hopf algebras $M \to N \to H$ with composition 'zero' we do get the induced map from $H$ extensions to trivlal $M$ extensions.

So what about the rest - not known yet (at least by us, it is quite possible that there is more information out there). It will almost certainly depend on the cocommutativity* of $H$, and likely on its commutativity.

*The classical construction depends on abelian groups. Extending topological cohomology beyond $H^1$ to nonabelian coefficients is a major problem.

# The binary operation on $H$ Hopf Galois extensions

Yokogawa (Osaka J. Math 18, 1981) proved that cocommutative Hopf Galois extensions over a commutative algebra form an abelian group.

The noncommutative case should be simple - For $H$ and $K$ extensions $P$ and $Q$ respectively, take $P \otimes_A Q$, take the invariant parts under the tensor product $H$ coaction, and then take the $H$ coaction on the $Q$ factor of the result. The problem is that the noncommutativity of the algebra gets in the way.

However we have seen cases that do work - the function algebras on the circle and $\mathbb{Z}/2$ - so there *ought* to be conditions to make this work.

# Further problems

1.1) What is needed for the general Thom construction? Presumably some sort of symmetrisation on $E \otimes_A E$ is necessary. Bimodule covariant derivatives on $\Omega^1 A$ are a source of such things, but are there others?

2.1) Find out what has been done before - the complication being that several subject areas are involved. However if much more is buried in the literature, it needs more publicity!

2.2) How many Hopf algebras make sensible coefficients (apart from the function algebras on the circle and $\mathbb{Z}/2$).

2.3) Just how much of classical $\mathbb{Z}/p$ cohomology (a very extensive theory) does extend to the noncommutative case?

2.4) Are there sensible (i.e. in principle calculable) $H^{>1}$s?

2.5) What is the ‘fundamental group’ of an algebra?