

Pseudodifferential Calculus for Singular Foliations

(Ongoing project with Georges Skandalis)

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Questions

Understand Laplacian + its spectral theory for

- 1 Foliations (possibly singular) on M : compact manifolds.
- 2 Flows on locally compact spaces (Dynamical systems).

X_1, \dots, X_n vector fields, $\mathcal{F} = \langle X_1, \dots, X_n \rangle$ is $[\cdot, \cdot]$ -stable.

Laplacian: $\Delta = -\sum X_i^2$.

- Show that Δ is essentially self adjoint on $L^2(M)$ and $L^2(\text{leaf})$.
- Where does the **resolvent** $(I + \Delta)^{-1}$ live?
- Where does the **index** of Δ , $(I + \Delta)^{-1}$ live?

In \mathbb{R}^n : $(I + \Delta)^{-1}$ **pseudodifferential** operator.

Δ **unbounded** (pseudo)differential operator.

Singular foliations

Definition

A **foliation** \mathcal{F} is a $C^\infty(M)$ -submodule of ΓTM such that

- 1 \forall open $U \subset M$, $C^\infty(U)\mathcal{F}$ is finitely generated
- 2 $X, Y \in \mathcal{F} \Rightarrow [X, Y] \in \mathcal{F}$

Definition

A **Stefan-Sussmann foliation** on M is a partition to connected submanifolds associated with an \mathcal{F} as above.

Examples

- 1 Regular foliations: definitions coincide. Only one choice:
 \mathcal{F} = tangent to leaves.
- 2 \mathbb{R} foliated by 3 leaves: $(-\infty, 0), \{0\}, (0, +\infty)$.
 $\mathcal{F} = \langle x^n \frac{\partial}{\partial x} \rangle$. Different foliation for different n .
- 3 \mathbb{R}^2 foliated by 2 leaves: $\{0\}, \mathbb{R}^2 \setminus \{0\}$. No best choice.
 \mathcal{F} given by action of $GL(2), SL(2), \mathbb{C}^*$.
- 4 Two notions of fiber:
 - $F_x = \text{ev}_x(\mathcal{F})$, where $\text{ev}_x : \mathcal{F} \rightarrow T_x M$
 - $\mathcal{F}_x = \mathcal{F}/I_x \mathcal{F}$, where $I_x =$ functions vanishing at x
 - \mathcal{F} : regular $\Rightarrow F_x = \mathcal{F}_x$.
 - In example 3: $\mathcal{F}_0 = \mathfrak{g}$ and $\mathcal{F}_x = F_x = \mathbb{R}^2$ for $x \neq 0$.
 -

$$0 \rightarrow \mathfrak{g}_x \rightarrow \mathcal{F}_x \xrightarrow{\text{ev}_x} F_x \rightarrow 0$$

Regular case

- Charts of (M, F) are $U \times T$. U : longitudinal, T : transverse. Transitions $f(u, t) = (g(u, t), h(t))$.
- A **holonomy** is a diffeomorphism $h: T \rightarrow T'$ s.t. t and $h(t)$ in same leaf. Get a *pseudo-group*.
- **Lie groupoid** with smallest isotropy:

$$\text{Hol}(F) = \{(x, y, \langle h(\gamma) \rangle) : x, y \in \text{same leaf}\}$$

- Construct $C^*(\text{Hol}(F))$. (Similar to Lie group C^* -alg.)
- Ψ DOs: **Families** $\{P_x : C_c^\infty(s^{-1}(x)) \rightarrow C^\infty(s^{-1}(x))\}_{x \in M}$ s.t.

$$U_g P_{s(g)} = P_{t(g)} U_g$$

Theorem (Monthubert, Pierrot, Vassout)

- ① Negative order: Elements of $C^*(\text{Hol}(F))$.
- ② Zero order: Bounded multipliers of $C^*(\text{Hol}(F))$.
- ③ Positive + elliptic: Unbounded multipliers of $C^*(\text{Hol}(F))$.

$$0 \rightarrow C^*(\text{Hol}(F)) \rightarrow \overline{\Psi^0(F)} \xrightarrow{\sigma} C_0(SF^*) \rightarrow 0$$

Symbol class

Relative K-theory: $\phi : A \rightarrow B$ homomorphism of unital C^* -algebras. $K_0(\phi)$ has generators (e^+, e^-, u) :

- $e^+, e^- \in M_n(A)$ idempotents,
- $uv = \phi(e^+), vu = \phi(e^-)$ for some $v \in M_n(B)$.

(Then divide by trivial triples and homotopy...)

e.g. $p : SF^* \rightarrow M$; **Mapping cone** of $p : C_0(M) \rightarrow C_0(SF^*)$:

$$\mathcal{C}_p = \{(f, \alpha) \in C_0([0, 1], C_0(SF^*)) \times C_0(M); p(\alpha) = f(0)\}$$

Turns out $\mathcal{C}_p = C_0(F^*)$, hence

$$K_0(p) = K_0(\mathcal{C}_p) = K_0(C_0(F^*))$$

Symbol of $P \in \Psi^0(F)$ is given by

- bundles $E^+, E^- \rightarrow M$
- isomorphism $\alpha : p^!(E^+) \rightarrow p^!(E^-)$, where $p : SF^* \rightarrow M$.

Whence $[\sigma(P)] \in K_0(p) = K_0(C_0(F^*))$.

Analytic index

Every $f \in C_0(M)$ is 0-order psdo acting on $C_0(\text{Hol}(F))$ by multiplication:

$$\begin{array}{ccc} C_0(M) & \xrightarrow{p} & C_0(SF^*) \\ \downarrow m & & \downarrow \text{Id} \\ 0 \rightarrow C^*(\text{Hol}(F)) & \longrightarrow & \overline{\Psi^0(F)} \xrightarrow{\sigma} C_0(SF^*) \longrightarrow 0 \end{array} \quad (1)$$

- Functoriality of mapping cone: $\varphi : \mathcal{C}_p \sim C_0(F^*) \rightarrow \mathcal{C}_\sigma$.
- Symbol map induces

$$0 \rightarrow C^*(\text{Hol}(F)) \xrightarrow{e} \mathcal{C}_\sigma \longrightarrow \mathcal{C}_{C_0(SF^*)} \rightarrow 0$$

6-term ex. seq.: $K_0(\mathcal{C}_\sigma) = K_0(C^*(\text{Hol}(F)))$.

- $[e] : K_0(C^*(\text{Hol}(F))) \rightarrow K_0(\mathcal{C}_\sigma)$ **invertible** because σ admits completely positive section.

$$\text{Ind}_{\text{an}} = [e]^{-1} \circ [\varphi] : K_0(C_0(F^*)) \longrightarrow K_0(C^*(\text{Hol}(F)))$$

Singular case

- No transversals!
- Take germs of local diffeomorphisms preserving the foliation. Problems:
 - ① Have to take a quotient. In regular case, only transverse action matters...
 - ② Topology?

Pieces of a Lie groupoid

$G \rightarrow M$ Lie groupoid, $U \subseteq M$ open at $x \in U$.

Choose ξ_1, \dots, ξ_n local sections of $A_U G$.

\exists nhd $W \subseteq G$ at 1_x , open ball $B^n \subseteq \mathbb{R}^n$, diffeom.

$$U \times B^n \rightarrow W, \quad (y, \theta_1, \dots, \theta_n) \mapsto \exp\left(\sum \theta_i \xi_i\right)(1_y)$$

Submersions $s, t : U \times B^n \rightarrow M$ where $s = \text{pr}_1$ and

$$t(y, \theta_1, \dots, \theta_n) \mapsto \exp\left(\sum \theta_i \rho(\xi_i)\right)(y)$$

where $\rho : AG \rightarrow TM$ anchor map.

Properties

- $U \times B^n \rightarrow W$ commutes with s, t
- Triple $(U \times B^n, t, s)$ satisfies

$$s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F}) = C_c^\infty(U \times B^n; \ker ds) + C_c^\infty(U \times B^n; \ker dt)$$

Bi-submersions

Now start directly from (M, \mathcal{F}) (no groupoid around). Recall

$$\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$$

Suppose $X_1, \dots, X_n \in \mathcal{F}$ define base of \mathcal{F}_x .

Let U neighborhood of $(x, 0) \in M \times \mathbb{R}^n$ domain of

$$t(y, \vec{\theta}) = \exp\left(\sum \theta_i X_i\right)(y)$$

Put $s = pr_1$. Then (U, t, s) satisfies

$$s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F}) = C_c^\infty(U; \ker ds) + C_c^\infty(U; \ker dt)$$

Definition

- (U, t, s) is a **bi-submersion** of (M, \mathcal{F}) at x .
(Obtain $\{(U_i, t_i, s_i)\}_{i \in I}$ such that $\cup_{i \in I} s_i(U_i) = M$.)

Examples of bi-submersions

- ① A Lie groupoid G is a bi-submersion
- ② If (\mathbf{U}, t, s) is a bi-submersion then the **inverse** (\mathbf{U}, s, t) is a bi-submersion.
- ③ **Composition** of bi-submersions: $\mathbf{U}_1 \times_{s_1, t_2} \mathbf{U}_2$.

Comparison of bi-submersions

Definition

Bisection of (U, t, s) : closed submanifold V s.t. $s|_V$ and $t|_V$ are local diffeomorphisms.

(Just take V s.t. $T_u U = T_u V \oplus \ker d_u s = T_u V \oplus \ker d_u t$.)

- $\phi = t|_V \circ (s|_V)^{-1}$ local diffeomorphism, $\phi^*(\mathcal{F}) \subseteq \mathcal{F}$.
Say **U carries ϕ at u** .
- U^{-1} carries ϕ^{-1} .
- $U_1 \circ U_2$ carries $\phi_1 \circ \phi_2$.
- If ψ is a local diffeomorphism then $(U, \psi^{-1} \circ t, s)$ carries ψ .

Theorem

(U, t, s) (W, t, s) bi-submersions, $u \in U$, $w \in W$. Then:

u, w carry **same ϕ** $\Leftrightarrow \exists$ **morphism** $f : U \rightarrow W$ with **$f(u) = w$** .

The holonomy groupoid

Definition

An **atlas** is a family $\mathcal{U} = (\mathcal{U}_i, t_i, s_i)$ covering M , stable under inversion and composition.

- 1 Groupoid $G_{\mathcal{U}}$ is the quotient of $\cup_{i \in I} \mathcal{U}_i$.
- 2 **Holonomy groupoid** of \mathcal{F} is the one associated with the minimal atlas.

Quotient topology on $G_{\mathcal{U}}$, usually bad. Non-Hausdorff in most cases. Different fibers have different dimensions.

e.g. action of $SL(2)$ on \mathbb{R}^2 .

Holonomy groupoid is a quotient of the **action groupoid**. It's

$$0 \times SL(2) \cup (\mathbb{R}^2 \setminus \{0\})^2$$

$\text{Hol}(\mathcal{F})$ smooth when \mathcal{F} is projective (almost regular).

The convolution algebra

Fix atlas $\mathcal{U} = \{U_i\}_{i \in I}$. Our algebra is **quotient** of

$$\bigoplus_{i \in I} C_c^\infty(U_i; \Omega^{1/2}U_i)$$

If M oriented we integrate sections of $\Lambda^n TM$. These transform using the Jacobian $J\phi$ of a diffeomorphism ϕ .

In non-oriented case we integrate objects that transform under $|J\phi|$. These are **1-densities**.

Definition

E vector space of dimension k and $\alpha \in \mathbb{R}$. Real α -density is map $f: \Lambda^k(E) \rightarrow \mathbb{R}$ s. t. $f(s\omega) = |s|^\alpha f(\omega)$ for $s \in \mathbb{R}$.

Half-densities on bi-submersions

Take (U, t, s) . $\Omega^{1/2}U = \frac{1}{2}$ -densities of bundle $\ker ds \times \ker dt \rightarrow U$.

$\Omega^{1/2}U$ line bundle – trivialized by any section.

Involution-Convolution

$\kappa_U : U^{-1} \rightarrow U$ the identity. Put $\Omega^{1/2}U^{-1} \equiv \kappa_U^*(\Omega^{1/2}U)$.

Involution

$C_c^\infty(U; \Omega^{1/2}U) \ni f \mapsto f^* = \overline{f \circ \kappa_U} \in C_c^\infty(U^{-1}; \Omega^{1/2}U^{-1})$.

We find $(\Omega^{1/2}(U \circ V))_{(u,v)} = (\Omega^{1/2}U)_u \otimes (\Omega^{1/2}V)_v$.

Convolution

$f \odot g \in C_c^\infty(U \circ V; \Omega^{1/2}U \circ V) \quad (u, v) \mapsto f(u) \otimes g(v)$.

The C^* -algebra

Proposition

If $(U_1, t_1, s_1), (U_2, t_2, s_2)$ bi-submersions, there is (W, t_W, s_W) and morphisms $p_i : U_i \rightarrow W$ which are **submersions**.

Let $p : U \rightarrow W$ submersion. Find

$$\Omega^{1/2}(U) = \Omega^1 \ker dp \oplus p_!(\Omega^{1/2}W)$$

Integration along fibers of p :

$$p_! : C_c^\infty(U; \Omega^{1/2}U) \rightarrow C_c^\infty(W; \Omega^{1/2}W)$$

\mathcal{A}_U the quotient by $f \sim p_!f$. Inherits involution, convolution.

Quotient map $Q_U : C_c^\infty(U; \Omega^{1/2}U) \rightarrow \mathcal{A}_U$.

Complete \mathcal{A}_U to $C^*(\mathcal{F})$ and $C_r^*(\mathcal{F})$. Get $*$ -homomorphism

$$\theta : \mathcal{A}_U \rightarrow C^*(\mathcal{F})$$

Desintegration

$\Pi : C^*(\mathcal{F}) \rightarrow B(H)$ non-degenerate repr. Extend to $\mathcal{M}(C^*(\mathcal{F}))$.

Fact: $C_0(M) \subseteq \mathcal{M}(C^*(\mathcal{F}))$ by

$$hQ_U(f) = Q_U((h \circ t)f), \quad Q_U(f)h = Q_U(f(h \circ s))$$

Restriction $\Pi : C_0(M) \rightarrow B(H)$ gives triple (μ, \mathcal{H}, π) where

- 1 μ **quasi-invariant** measure class on M ;
- 2 $\mathcal{H} = \{\mathcal{H}_x\}_{x \in M}$ measurable field of Hilbert spaces s.t.

$$H = \int^{\oplus} \mathcal{H}_x d\mu(x)$$

- 3 Measurable morphism of groupoids

$$\pi : \text{Hol}(\mathcal{F}) \rightarrow \mathcal{H}$$

Theorem

Repr. of $C^*(\mathcal{F})$ to Hilbert space H correspond to repr. (μ, \mathcal{H}, π) of $\text{Hol}(\mathcal{F})$ to Hilbert bundle \mathcal{H} .

Observation

Submfd $V \subseteq M$, vector field X , distribution

$$q_X : f \mapsto \int_V Xf$$

If X tangent to V ,

$$\int_V Xf = - \int_V \operatorname{div}(X) \cdot f$$

So up to zero order, q_X depends on image of X in NV .

The space $P(U, V)$

(U, t, s) bi-subm., $V \subset U$ identity bisection, $N \rightarrow V$ normal bundle. Symbol $\alpha \in S_{cl, c}^m(V, N^*; \Omega^1 N^*)$.

$C^\infty(V)$ -linear $P_\alpha : C_c^\infty(N; \Omega^1 N) \rightarrow C^\infty(V)$:

$$\langle P_\alpha, f \rangle(x) = (2\pi)^{-k} \int_{N_x^* \times N_x} \alpha(x, \xi) e^{-i\langle u, \xi \rangle} f(u)$$

Integrating on V gives distribution. P_α pseudodifferential kernel.

Definition

$P^m(U, V) =$ Generalized functions on U with pseudodifferential singularities on V :

$$P = h + \chi \cdot P_\alpha \circ \phi$$

- $h \in C^\infty(U)$, $\phi : U_1 \rightarrow N$ tubular neighborhood;
- χ smooth “bump function” s.t. $\chi|_V = 1$, $\chi|_{U_1^c} = 0$

e.g. $q_\chi : f \mapsto \int_V \chi f$ is in $P^1(U, V)$.

Properties of $\mathcal{P}(\mathbf{U}, \mathbf{V})$

- 1 $C_c^\infty(\mathbf{U})$ is dense in $\mathcal{P}(\mathbf{U}, \mathbf{V})$.
- 2 Symbol:

$$0 \rightarrow \mathcal{P}^{m-1}(\mathbf{U}, \mathbf{V}) \rightarrow \mathcal{P}^m(\mathbf{U}, \mathbf{V}) \xrightarrow{\sigma_m} C^\infty(\mathbf{S}^*\mathbf{N}) \rightarrow 0$$

If $x \in \mathbf{V}$, $\xi \in \mathbf{N}_x^*$, $\xi \neq 0$. Let $\mathbf{U} = \mathbf{N}$. Then \mathcal{P} is a distribution $C_c^\infty(\mathbf{N}) \rightarrow C_c^\infty(\mathbf{V})$.

Put

$$\sigma_m(x, \xi) = \lim_{\tau \rightarrow +\infty} (i\tau)^{-m} \mathcal{P}(e^{i\tau\phi})(x)$$

- $\phi \in C_c^\infty(\mathbf{N})$ with derivative ξ at $x \in \mathbf{V}$.
- $\chi \in C_c^\infty(\mathbf{N})$ equal to 1 in a nhd of x .

e.g. $\sigma_1(q_X) : \xi \mapsto i \langle X | \xi \rangle$

Putting $\mathcal{P}(\mathbf{U}, \mathbf{V})$ in $C^*(\mathcal{F})$

Recall $\theta : \mathcal{A}_{\mathbf{U}} \rightarrow C^*(\mathcal{F})$. Given (\mathbf{U}, t, s) let $\theta_{\mathbf{U}}$ the composition:

$$\begin{array}{ccc} C_c^\infty(\mathbf{U}; \Omega^{1/2}) & \xrightarrow{Q_{\mathbf{U}}} & \mathcal{A}_{\mathbf{U}} \\ & \searrow \theta_{\mathbf{U}} & \downarrow \theta \\ & & C^*(\mathcal{F}) \end{array} \quad (2)$$

Take $\Pi : C^*(\mathcal{F}) \rightarrow B(\mathcal{H})$ non-degenerate. Put $\Pi_{\mathbf{U}} = \Pi \circ \theta_{\mathbf{U}}$.

Using **desintegration** we prove:

Theorem

- 1 Π extends to $\Pi_{\mathbf{U}, \mathbf{V}} : P_c(\mathbf{U}, \mathbf{V}; \Omega^{1/2}) \rightarrow \mathcal{L}(\mathcal{H})$ (\mathcal{H} : Hilbert bundle), s.t. $\Pi_{\mathbf{U}, \mathbf{V}}(Q) = \Pi_W(p_!(Q))$ for any submersion $p : W \rightarrow \mathbf{U}$.
- 2 If Π is faithful, there is a linear map

$$\theta_{\mathbf{U}, \mathbf{V}} : P_c(\mathbf{U}, \mathbf{V}; \Omega^{1/2}) \rightarrow \theta(\mathcal{A}_{\mathbf{U}})$$

$$\text{s.t. } \theta_{\mathbf{U}, \mathbf{V}}(Q) = \theta_W(p_!(Q)).$$

$P(\mathcal{U}, \mathcal{V})$ as multipliers of $C^*(\mathcal{F})$

$(\mathcal{U}, t, s), (\mathcal{U}', t', s')$, projection $p_1 : \mathcal{U} \circ \mathcal{U}' \rightarrow \mathcal{U}$.

$P \in P_c(\mathcal{U}, \mathcal{V}; \Omega^{1/2})$ and $f \in C_c^\infty(\mathcal{U}'; \Omega^{1/2})$. Define

$$P * f = (p_1)_!(P) \cdot f$$

Proposition

There is a multiplier $\tilde{\theta}_{\mathcal{U}, \mathcal{V}}(P)$ of $\theta(\mathcal{A}_{\mathcal{U}})$ such that

- $\tilde{\theta}_{\mathcal{U}, \mathcal{V}}(P)\theta_{\mathcal{U}'}(f) = \theta_{\mathcal{U} \circ \mathcal{U}', \mathcal{V} \circ \mathcal{U}'}(P * f)$.
- $\theta_{\mathcal{U}''}(g)\tilde{\theta}_{\mathcal{U}, \mathcal{V}}(P) = \theta_{\mathcal{U} \circ \mathcal{U}', \mathcal{U}'' \circ \mathcal{V}}(g * P)$.

Definition

$\mathcal{U} = \{(\mathcal{U}_i, t_i, s_i)\}_{i \in I}$ atlas, $\mathcal{V} = \{\mathcal{V}_i\}_{i \in I}$ identity bisections.

$\Psi_c^m(\mathcal{U}, \mathcal{V})$ is image in $\mathcal{M}(\theta(\mathcal{A}_{\mathcal{U}}))$ of $\bigoplus_{i \in I} P(\mathcal{U}_i, \mathcal{V}_i; \Omega^{1,2})$.

Just constructed linear map

$$\tilde{\theta}_{\mathcal{U}, \mathcal{V}} : P_c(\mathcal{U}, \mathcal{V}; \Omega^{1/2}) \rightarrow \Psi_c^m(\mathcal{U}, \mathcal{V})$$

Principal symbol

$\mathcal{F}^* = \cup_{x \in M} \mathcal{F}_x^*$: nice locally compact space.

e.g. $SO(3)$ acting on \mathbb{R}^3 : $\mathcal{F}^* = \cup_{\xi \in \mathbb{R}^3} \{x \in \mathbb{R}^3 \mid \langle x, \xi \rangle = 0\}$

Definition

- 1 $P_i \in P_c^m(\mathcal{U}_i, \mathcal{V}_i)$: $\tilde{\sigma}_m(P_i) = \sigma_m(P_i)$ restr. to $\mathcal{F}^*/M \subseteq N^*/V$.
- 2 $P \in \oplus_{i \in I} P_c^m(\mathcal{U}_i, \mathcal{V}_i)$. Define $\tilde{\sigma}_m(P)$ extending by linearity...

Not clear if $\tilde{\sigma}_m$ defined in $\Psi^m(\mathcal{U}, \mathcal{V})$.

Yes if $\text{Hol}(\mathcal{F})$ **s-smooth**. Because then left regular repr. on $L^2(\text{Hol}(\mathcal{F})_x)$ is defined...

Not clear if $\exists \quad 0 \rightarrow \Psi_c^{m-1} \rightarrow \Psi_c^m \xrightarrow{\tilde{\sigma}_m} C_0(S\mathcal{F}^*) \rightarrow 0$

e.g. $SO(3)$ -action and 0-order symbol

$$\alpha(x, \xi) = \begin{cases} e^{-\frac{1}{\langle x, \xi \rangle^2}} & \text{out of } \mathcal{F}^* \\ 0 & \text{in } \mathcal{F}^* \end{cases}$$

\nexists order -1 operator with symbol α around \mathcal{F}^* !

The space $\Psi^\infty(\mathcal{U}, \mathcal{V})$

- ① Subalgebra of $\mathcal{M}(\theta(\mathcal{A}_U))$: Given $P_i \in P^{m_i}$, there is $P \in P^{m_1+m_2}$ s.t.:

$$\tilde{\theta}(P_1)\tilde{\theta}(P_2) = \tilde{\theta}(P) \quad \text{and} \quad \tilde{\sigma}_{m_1}(P_1)\tilde{\sigma}_{m_2}(P_2) = \tilde{\sigma}_{m_1+m_2}(P)$$

- ② If $P \in P^m$ **elliptic** then $\exists Q \in P^{-m}$ s.t.
 $1 - \tilde{\theta}(P)\tilde{\theta}(Q), \quad 1 - \tilde{\theta}(Q)\tilde{\theta}(P)$ regularizing.

- ③ \exists square roots...

- ④ \exists quotient B of $C_0(S^*\mathcal{F})$ s.t.

$$0 \rightarrow C^*(M, \mathcal{F}) \rightarrow \overline{\Psi^0(M; \mathcal{F})} \xrightarrow{\sigma} B \rightarrow 0$$

- ⑤
- $\text{Ord}(P) < 0 \Rightarrow P \in C^*(M, \mathcal{F})$;
 - $\text{Ord}(P) = 0 \Rightarrow P$: bounded multiplier of $C^*(M, \mathcal{F})$;
 - $\text{Ord}(P) > 0, P$: elliptic $\Rightarrow P$: unbounded multiplier of $C^*(M, \mathcal{F})$ (regular).

Conclusions

- Formally self-adjoint (pseudo)differential operators are essentially self-adjoint in **every** representation. In particular, in $L^2(\text{leaf}), L^2(M)$.
- Have same spectrum in **every** faithful representation.

Conclusions

- 1 Laplacian Δ is unbounded operator in $L^2(M)$.
- 2 $C^*(\mathcal{F})$ and our pseudodifferential calculus are suited for index problems.

In particular, analytic index $\text{Ind}_{\text{an}} : K_0(C_0(\mathcal{F}^*)) \rightarrow K_0(C^*(\mathcal{F}))$ is defined by relative K-theory...

Tangent foliation

(M, \mathcal{F}) with leaves $L_x, x \in M$. Foliate $M \times \mathbb{R}$ with leaves:

- $\hat{L}_{(x,t)} = L_x \times \{t\}$ if $t \neq 0$
- $\hat{L}_{(x,0)} = \{x\} \times \{0\}$

$C^\infty(M \times \mathbb{R})$ -module $T\mathcal{F}$ generated by $\mathcal{F} \otimes \text{pr}_2$.

$T\mathcal{F} = \text{fin. sums } \sum f_i(X_i \otimes 1), \quad X_i \in \mathcal{F}, \quad f_i \in \text{pr}_2 \cdot C_c^\infty(M \times \mathbb{R})$

Bi-submersions: Consider

- (U, t, s) bi-submersion of (M, \mathcal{F}) ;
- $V \subseteq U$ identity bisection.

normal cone deformation

$$\tilde{U} = U \times \mathbb{R}^* \amalg NV \times \{0\}$$

Groupoid:

$$\text{Hol}(T\mathcal{F}) = (U_{x \in M} \mathcal{F}_x) \times \{0\} \amalg \text{Hol}(\mathcal{F}) \times \mathbb{R}^*$$

Analytic index via tangent foliation

Restricting to $[0, 1]$ we get:

$$0 \rightarrow C_0((0, 1]) \otimes C^*(M; \mathcal{F}) \rightarrow C^*(M \times [0, 1]; T\mathcal{F}) \xrightarrow{ev_0} C_0(\mathcal{F}^*) \rightarrow 0$$

- $C_0((0, 1])$ contractible, so ev_0 invertible in K-theory.
- Also $ev_1 : C^*(M \times \mathbb{R}; T\mathcal{F}) \rightarrow C^*(M; \mathcal{F})$.

Theorem

$$\text{Ind}_{\text{an}} = [ev_1] \circ [ev_0]^{-1} : K_0(\mathcal{F}^*) \longrightarrow K_0(C^*(\mathcal{F}))$$

Papers

IA and Georges Skandalis

[1] The analytic index of elliptic pseudodifferential operators on a singular foliation. (Submitted.)

arxiv:1004.3797

[2] Pseudodifferential calculus for singular foliations. (To appear in *J. Noncomm. Geom.*)

arxiv:0909.1342

[3] The holonomy groupoid of a singular foliation. *J. Reine Angew. Math.* 626(2009), 1–37.

arxiv:math/0612370