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A Higher Dimensional Stationary Rotating Black Hole Must be Axisymmetric

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Abstract

A key result in the proof of black hole uniqueness in 4-dimensions is that a stationary black hole that is "rotating"—i.e., is such that the stationary Killing field is not everywhere normal to the horizon—must be axisymmetric. The proof of this result in 4-dimensions relies on the fact that the orbits of the stationary Killing field on the horizon have the property that they must return to the same null geodesic generator of the horizon after a certain period, P. This latter property follows, in turn, from the fact that the cross-sections of the horizon are two-dimensional spheres. However, in spacetimes of dimension greater than 4, it is no longer true that the orbits of the stationary Killing field on the horizon must return to the same null geodesic generator. In this paper, we prove that, nevertheless, a higher dimensional stationary black hole that is rotating must be axisymmetric. No assumptions are made concerning the topology of the horizon cross-sections other than that they are compact. However, we assume that the horizon is non-degenerate and, as in the 4-dimensional proof, that the spacetime is analytic.

1 Introduction

Consider an n-dimensional stationary spacetime containing a black hole. Since the event horizon of the black hole must be mapped into itself by the action of any isometry, the asymptotically timelike Killing field t^a must be tangent to the horizon. Therefore, we have two cases to consider: (i) t^a is normal to the horizon, i.e., tangent to the null geodesic generators of the horizon; (ii) t^a is not normal to the horizon. In 4-dimensions it is known

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that in case (i), for suitably regular non-extremal vacuum or Einstein-Maxwell black holes, the black hole must be static [42, 5]. Furthermore, in 4-dimensions it is known that in case (ii), under fairly general assumptions about the nature of the matter content but assuming analyticity of the spacetime and non-extremality of the black hole, there must exist an additional Killing field that is normal to the horizon. It can then be shown that the black hole must be axisymmetric[1] as well as stationary [18, 19]. This latter result is often referred to as a "rigidity theorem," since it implies that the horizon generators of a "rotating" black hole (i.e., a black hole for which t^a is not normal to the horizon) must rotate rigidly with respect to infinity. A proof of the rigidity theorem in 4-dimensions which partially eliminates the analyticity assumption was given by Friedrich, Racz, and Wald [9, 32], based upon an argument of Isenberg and Moncrief [27, 20] concerning the properties of spacetimes with a compact null surface with closed generators. The above results for both cases (i) and (ii) are critical steps in the proofs of black hole uniqueness in 4-dimensions, since they allow one to apply Israel's theorems [23, 24] in case (i) and the Carter-Robinson-Mazur-Bunting theorems [2, 36, 25, 1] in case (ii).

Many attempts to unify the forces and equations of nature involve the consideration of spacetimes with $n > 4$ dimensions. Therefore, it is of considerable interest to consider a generalization of the rigidity theorem to higher dimensions, especially in view of the fact that there seems to be a larger variety of black hole solutions (see e.g., [7, 12, 15]), the classification of which has not been achieved yet. [2] The purpose of this paper is to present a proof of the rigidity theorem in higher dimensions for non-extremal black holes.

The dimensionality of the spacetime enters the proof of the rigidity theorem in 4-dimensions in the following key way: The expansion and shear of the null geodesic generators of the horizon of a stationary black hole can be shown to vanish (see below). The induced (degenerate) metric on the $(n-1)$-dimensional horizon gives rise to a Riemannian metric, γ_{ab}, on an arbitrary $(n-2)$-dimensional cross-section, Σ, of the horizon. On account of the vanishing shear and expansion, all cross-sections of the horizon are isometric, and the projection of the stationary Killing field t^a onto Σ gives rise to a Killing field, s^a, of γ_{ab} on Σ. In case (ii), s^a does not vanish identically. Now, when $n = 4$, it is known that Σ must have the topology of a 2-sphere, S^2. Since the Euler characteristic of S^2 is nonzero, it follows that s^a must vanish at some point $p \in \Sigma$. However, since Σ is 2-dimensional, it then follows that the isometries generated by s^a simply rotate the tangent space at p. It then follows that all of the orbits of s^a are periodic with a fixed period P, from which it follows that, after period P, the orbits of t^a on the horizon must return to the same generator. Consequently, if we identify points in spacetime that differ by the action of the stationary isometry of parameter P, the horizon becomes a compact null surface with closed null geodesic generators. The theorem of Isenberg and Moncrief [27, 20] then provides the desired additional Killing field normal to this null surface.

In $n > 4$ dimensions, the Euler characteristic of Σ may vanish, and, even if it is non-vanishing, if $n > 5$ there is no reason that the isometries generated by s^a need have closed orbits even when s^a vanishes at some point $p \in \Sigma$. Thus, for example, even in the 5-dimensional Myers-Perry black hole solution [30] with cross section topology $\Sigma = S^3$, one can choose the rotational parameters of the solution so that the orbits of the stationary

[1] In this paper, by "axisymmetric" we mean that spacetime possesses one-parameter group of isometries isomorphic to $U(1)$ whose orbits are spacelike. We do not require that the Killing field vanishes on an "axis."

[2] There have recently appeared several works on general properties of a class of stationary, axisymmetric vacuum solutions, including an n-dimensional generalization of the Weyl solutions for the static case (see e.g., [3, 6, 16, 17], and see also [26, 43] and references therein for some techniques of generating such solutions in 5-dimensions).

Killing field t^a do not map horizon generators into themselves.

One possible approach to generalizing the rigidity theorem to higher dimensions would be to choose an arbitrary $P > 0$ and identify points in the spacetime that differ by the action of the stationary isometry of parameter P. Under this identification, the horizon would again become a compact null surface, but now its null geodesic generators would no longer be closed. The rigidity theorem would follow if the results of [27, 20] could be generalized to the case of compact null surfaces that are ruled by non-closed generators. We have learned that Isenberg and Moncrief are presently working on such a generalization [22], so it is possible that the rigidity theorem can be proven in this way.

However, we shall not proceed in this manner, but rather will parallel the steps of [27, 20], replacing arguments that rely on the presence of closed null generators with arguments that rely on the presence of stationary isometries. Since on the horizon we may write

$$t^a = n^a + s^a \,, \tag{1}$$

where n^a is tangent to the null geodesic generators and s^a is tangent to cross-sections of the horizon, the stationarity in essence allows us to replace Lie derivatives with respect to n^a by Lie derivatives with respect to s^a. Thus, equations in [27, 20] that can be solved by integrating quantities along the orbits of the closed null geodesics correspond here to equations that can be solved if one can suitably integrate these equations along the orbits of s^a in Σ. Although the orbits of s^a are not closed in general, we can appeal to basic results of ergodic theory together with the fact that s^a generates isometries of Σ to solve these equations.

For simplicity, we will focus attention on the vacuum Einstein's equation, but we will indicate in section 4 how our proofs can be extended to models with a cosmological constant and a Maxwell field. As in [18, 19] and in [27, 20], we will assume analyticity, but we shall indicate how this assumption can be partially removed (to prove existence of a Killing field *inside* the black hole) by arguments similar to those given in [9, 32]. The non-extremality condition is used for certain constructions in the proof (as well as in the arguments partially removing the analyticity condition), and it would not appear to be straightforward to generalize our arguments to remove this restriction when the orbits of s^a are not closed.

Our signature convention for g_{ab} is $(-,+,+,\cdots)$. We define the Riemann tensor by $R_{abc}{}^d k_d = 2\nabla_{[a}\nabla_{b]}k_c$ and the Ricci tensor by $R_{ab} = R_{acb}{}^c$. We also set $8\pi G = 1$.

2 Proof of existence of a horizon Killing field

Let (M, g_{ab}) be an n-dimensional, smooth, asymptotically flat, stationary solution to the vacuum Einstein equation containing a black hole. Thus, we assume the existence in the spacetime of a Killing field t^a with complete orbits which are timelike near infinity. Let H denote a connected component of the portion of the event horizon of the black hole that lies to the future of $\mathscr{I}^-$. We assume that H has topology $\mathbb{R} \times \Sigma$, where Σ is compact. Following Isenberg and Moncrief [27, 20], our aim in this section is to prove that there exists a vector field K^a defined in a neighborhood of H which is normal to H and on H satisfies

$$\underbrace{\mathscr{L}_\ell \mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m \text{ times}}(\mathscr{L}_K g_{ab}) = 0, \quad m = 0, 1, 2, \ldots \,, \tag{2}$$

where ℓ is an arbitrary vector field transverse to H. As we shall show at the end of this section, if we assume analyticity of g_{ab} and of H it follows that K^a is a Killing field. We also will explain at the end of this section how to partially remove the assumption of analyticity of g_{ab} and H.

We shall proceed by constructing a candidate Killing field, K^a, and then proving that eq. (2) holds for K^a. This candidate Killing field is expected to satisfy the following properties: (i) K^a should be normal to H. (ii) If we define S^a by

$$S^a = t^a - K^a \tag{3}$$

then, on H, S^a should be tangent to cross-sections[3] of H. (iii) K^a should commute with t^a. (iv) K^a should have constant surface gravity on H, i.e., on H we should have $K^a \nabla_a K^b = \kappa K^b$ with κ constant on H, since, by the zeroth law of black hole mechanics, this property is known to hold on any Killing horizon in any vacuum solution of Einstein's equation.

We begin by choosing a cross-section Σ, of H. By arguments similar to those given in the proof of proposition 4.1 of [5], we may assume without loss of generality that Σ has been chosen so that each orbit of t^a on H intersects Σ at precisely one point, so that t^a is everywhere transverse to Σ. We extend Σ to a foliation, $\Sigma(u)$, of H by the action of the time translation isometries, i.e., we define $\Sigma(u) = \phi_u(\Sigma)$, where ϕ_u denotes the one-parameter group of isometries generated by t^a. Note that the function u on H that labels the cross-sections in this foliation automatically satisfies

$$\mathscr{L}_t u = 1\,. \tag{4}$$

Next, we define n^a and s^a on H by

$$t^a = n^a + s^a\,, \tag{5}$$

where n^a is normal to H and s^a is tangent to $\Sigma(u)$. It follows from the transversality of t^a that n^a is everywhere nonvanishing and future-directed. Note also that $\mathscr{L}_n u = 1$ on H. Our strategy is to extend this definition of n^a to a neighborhood of H via Gaussian null coordinates. This construction of n^a obviously satisfies conditions (i) and (ii) above, and it also will be shown below that it satisfies condition (iii). However, it will, in general, fail to satisfy (iv). We shall then modify our foliation so as to produce a new foliation $\tilde{\Sigma}(\tilde{u})$ so that (iv) holds as well. We will then show that the corresponding $K^a = \tilde{n}^a$ satisfies eq. (2).

Given our choice of $\Sigma(u)$ and the corresponding choice of n^a on H, we can uniquely define a past-directed null vector field ℓ^a on H by the requirements that $n^a \ell_a = 1$, and that ℓ^a is orthogonal to each $\Sigma(u)$. Let r denote the affine parameter on the null geodesics determined by ℓ^a, with $r = 0$ on H. Let $x^A = (x^1, x^2, \ldots, x^{n-2})$ be local coordinates on an open subset of Σ. Of course, it will take more than one coordinate patch to cover Σ, but there is no problem in patching together local results, so no harm is done in pretending that x^A covers Σ. We extend the coordinates x^A from Σ to H by demanding that they be constant along the orbits of n^a. We then extend u and x^A to a neighborhood of H by requiring these quantities to be constant along the orbits of ℓ^a. It is easily seen that the quantities (u, r, x^A) define coordinates covering a neighborhood of H. Coordinates that are constructed in this manner are known as *Gaussian null coordinates* and are unique up to the choice of Σ and the choice of coordinates x^A on Σ. It follows immediately that on H we have

$$n^a = \left(\frac{\partial}{\partial u}\right)^a, \quad \ell^a = \left(\frac{\partial}{\partial r}\right)^a, \tag{6}$$

[3]Note that as already mentioned above, since H is mapped into itself by the time translation isometries, t^a must be tangent to H, so S^a is automatically tangent to H. Condition (iii) requires that there exist a foliation of H by cross-sections $\Sigma(u)$ such that each orbit of S^a is contained in a single cross-section.

and we extend n^a and ℓ^a to a neighborhood of H by these formulas. Clearly, n^a and ℓ^a commute, since they are coordinate vector fields.

Note that we have

$$\ell^a \nabla_a (n_b \ell^b) = \ell^b \ell^a \nabla_a n_b = \ell^b n^a \nabla_a \ell_b = \frac{1}{2} n^a \nabla_a (\ell^b \ell_b) = 0 \,, \tag{7}$$

so $n_a \ell^a = 1$ everywhere, not just on H. Similarly, we have $\ell_a (\partial/\partial x^A)^a = 0$ everywhere. It follows that in Gaussian null coordinates, the metric in a neighborhood of H takes the form

$$g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = 2 \left(\mathrm{d}r - r\alpha \mathrm{d}u - r\beta_A \mathrm{d}x^A \right) \mathrm{d}u + \gamma_{AB} \mathrm{d}x^A \mathrm{d}x^B \,, \tag{8}$$

where, again, A is a labeling index that runs from 1 to $n-2$. We write

$$\beta_a = \beta_A (\mathrm{d}x^A)_a \,, \quad \gamma_{ab} = \gamma_{AB} (\mathrm{d}x^A)_a (\mathrm{d}x^B)_b \,. \tag{9}$$

Note that α, β_a, and γ_{ab} are independent of the choice of coordinates, x^A, and thus are globally defined in an open neighborhood of H. From the form of the metric, we clearly have $\beta_a n^a = \beta_a \ell^a = 0$ and $\gamma_{ab} n^a = \gamma_{ab} \ell^a = 0$. It then follows that $\gamma^a{}_b$ is the orthogonal projector onto the subspace of the tangent space perpendicular to n^a and ℓ^a, where here and elsewhere, all indices are raised and lowered with the spacetime metric g_{ab}. Note that when $r \neq 0$, i.e., off of the horizon, γ_{ab} differs from the metric q_{ab}, on the $(n-2)$-dimensional submanifolds, $\Sigma(u,r)$, of constant (u,r), since n^a fails to be perpendicular to these surfaces. Here, q_{ab} is defined by the condition that $q^a{}_b$ is the orthogonal projector onto the subspace of the tangent space that is tangent to $\Sigma(u,r)$; the relationship between γ_{ab} and q_{ab} is given by

$$q_{ab} = r^2 \beta^c \beta_c \ell_a \ell_b - 2r \beta_{(a} \ell_{b)} + \gamma_{ab} \,. \tag{10}$$

However, since on H (where $r=0$), we have $\gamma_{ab} = q_{ab}$, we will refer to γ_{ab} as the metric on the cross-sections $\Sigma(u)$ of H.

Thus, we see that in Gaussian null coordinates the spacetime metric, g_{ab}, is characterized by the quantities α, β_a, and γ_{ab}. In terms of these quantities, if we choose $K^a = n^a$, then the condition (2) will hold if and only if the conditions

$$\begin{aligned} \underbrace{\mathscr{L}_\ell \mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m \text{ times}} (\mathscr{L}_n \gamma_{ab}) &= 0 \,, \\ \underbrace{\mathscr{L}_\ell \mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m \text{ times}} (\mathscr{L}_n \alpha) &= 0 \,, \\ \underbrace{\mathscr{L}_\ell \mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m \text{ times}} (\mathscr{L}_n \beta_a) &= 0 \,, \end{aligned} \tag{11}$$

hold on H.

Since the vector fields n^a and ℓ^a are uniquely determined by the foliation $\Sigma(u)$ and since $\phi_u[\Sigma(u')] = \Sigma(u+u')$ (i.e., the time translations leave the foliation invariant), it follows immediately that n^a and ℓ^a are invariant under ϕ_u. Hence, we have $\mathscr{L}_t n^a = \mathscr{L}_t \ell^a = 0$, so, in particular, condition (iii) holds, as claimed above. Similarly, we have $\mathscr{L}_t r = 0$ and $\mathscr{L}_t u = 1$ throughout the region where the Gaussian null coordinates are defined. Since $\mathscr{L}_t g_{ab} = 0$, we obtain from eq. (8)

$$0 = -2r \mathscr{L}_t \alpha \, \nabla_a u \, \nabla_b u - 2r \, \mathscr{L}_t \beta_{(a} \nabla_{b)} u + \mathscr{L}_t \gamma_{ab} \,. \tag{12}$$

Contraction of this equation with $n^a n^b$ yields

$$\mathscr{L}_t \alpha = 0\,. \tag{13}$$

Contraction with n^a then yields

$$\mathscr{L}_t \beta_a = 0\,, \tag{14}$$

and we then also immediately obtain

$$\mathscr{L}_t \gamma_{ab} = 0\,. \tag{15}$$

The next step in the analysis is to use the Einstein equation $R_{ab}n^a n^b = 0$ on H, in a manner completely in parallel with the 4-dimensional case [19]. This equation is precisely the Raychaudhuri equation for the congruence of null curves defined by n^a on H. Since that congruence is twist-free on H, we obtain on H

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\theta = -\frac{1}{n-2}\theta^2 - \widehat{\sigma}_{ab}\widehat{\sigma}^{ab}\,, \tag{16}$$

where θ denotes the expansion of the null geodesic generators of H, $\widehat{\sigma}_{ab}$ denotes their shear, and λ is the affine parameter along null geodesic generators of H with tangent n^a. Now, by the same arguments as used to prove the area theorem [19], we cannot have $\theta < 0$ on H. On the other hand, the rate of change of the area, $A(u)$, of $\Sigma(u)$ (defined with respect to the metric $q_{ab} = \gamma_{ab}$) is given by

$$\frac{\mathrm{d}}{\mathrm{d}u}A(u) = \frac{1}{2}\int_\Sigma \left(\frac{\partial\lambda}{\partial u}\right)\theta\sqrt{\gamma}\,\mathrm{d}^{n-2}x\,. \tag{17}$$

However, since $\Sigma(u)$ is related to Σ by the isometry ϕ_u, the left side of this equation must vanish. Since $\partial\lambda/\partial u > 0$ on H, this shows that $\theta = 0$ on H. It then follows immediately that $\widehat{\sigma}_{ab} = 0$ on H. Now on H, the shear is equal to the trace free part of $\mathscr{L}_n\gamma_{ab}$ while the expansion is equal to the trace of this quantity. So we have shown that $\mathscr{L}_n\gamma_{ab} = 0$ on H. Thus, the first equation in eq. (11) holds with $m = 0$.

However, n^a in general fails to satisfy condition (iv) above. Indeed, from the form, eq. (8), of the metric, we see that the surface gravity, κ, associated with n^a is simply α, and there is no reason why α need be constant on H. Since $\mathscr{L}_n\gamma_{ab} = 0$ on H, the Einstein equation $R_{ab}n^a(\partial/\partial x^A)^b = 0$ on H yields

$$D_a\alpha = \frac{1}{2}\mathscr{L}_n\beta_a\,, \tag{18}$$

(see eq. (79) of Appendix A) where D_a denotes the derivative operator on $\Sigma(u)$, i.e., $D_a\alpha = q_a{}^b\nabla_b\alpha = \gamma_a{}^b\nabla_b\alpha$. Thus, if α is not constant on H, then the last equation in eq. (11) fails to hold even when $m = 0$. As previously indicated, our strategy is repair this problem by choosing a new cross-section $\tilde{\Sigma}$ so that the corresponding $\tilde{n}^a$ arising from the Gaussian null coordinate construction will have constant surface gravity on H. The determination of this $\tilde{\Sigma}$ requires some intermediate constructions, to which we now turn.

First, since we already know that $\mathscr{L}_t\gamma_{ab} = 0$ everywhere and that $\mathscr{L}_n\gamma_{ab} = 0$ on H, it follows immediately from the fact that $t^a = s^a + n^a$ on H that

$$\mathscr{L}_s\gamma_{ab} = 0\,, \tag{19}$$

on H (for any choice Σ). Thus, s^a is a Killing vector field for the Riemannian metric $\gamma_{ab} = q_{ab}$ on Σ. Therefore the flow, $\hat{\phi}_\tau : \Sigma \to \Sigma$ of s^a yields a one-parameter group of isometries of γ_{ab}, which coincides with the projection of the flow ϕ_u of the original Killing field t^a to Σ.

We define κ to be the mean value of α on Σ,

$$\kappa = \frac{1}{A(\Sigma)} \int_\Sigma \alpha \sqrt{\gamma}\, \mathrm{d}^{n-2}x\,, \tag{20}$$

where $A(\Sigma)$ is the area of Σ with respect to the metric γ_{ab}. In the following we will assume that $\kappa \neq 0$, i.e., that we are in the "non-degenerate case." Given that $\kappa \neq 0$, we may assume without loss of generality, that $\kappa > 0$.

We seek a new Gaussian null coordinate system $(\tilde{u}, \tilde{r}, \tilde{x}^A)$ satisfying all of the above properties of (u, r, x^A) together with the additional requirement that $\tilde{\alpha} = \kappa$, i.e., constancy of the surface gravity. We now determine the conditions that these new coordinates would have to satisfy. Since clearly $\tilde{n}^a$ must be proportional to n^a, we have

$$\tilde{n}^a = f n^a\,, \tag{21}$$

for some positive function f. Since $\mathscr{L}_t \tilde{n}^a = \mathscr{L}_t n^a = 0$, we must have $\mathscr{L}_t f = 0$. Since on H we have $n^a \nabla_a n^b = \alpha n^b$ and $\tilde{\alpha}$ is given by

$$\tilde{n}^a \nabla_a \tilde{n}^b = \tilde{\alpha} \tilde{n}^b\,, \tag{22}$$

we find that f must satisfy

$$\tilde{\alpha} = \mathscr{L}_n f + \alpha f = -\mathscr{L}_s f + \alpha f = \kappa\,. \tag{23}$$

The last equality provides an equation that must be satisfied by f on Σ. In order to establish that a solution to this equation exists, we first prove the following lemma:

Lemma 1 For any $x \in \Sigma$, we have

$$\kappa = \lim_{S\to\infty} \frac{1}{S} \int_0^S \alpha(\hat{\phi}_\tau(x))\, \mathrm{d}\tau\,. \tag{24}$$

Furthermore, the convergence of the limit is uniform in x. Similarly, x-derivatives of $S^{-1} \int_0^S \alpha(\hat{\phi}_\tau(x))\, \mathrm{d}\tau$ converge to 0 uniformly in x as $S \to \infty$.

Proof: The von Neumann ergodic theorem (see e.g.,[44]) states that if F is an L^p function for $1 \leq p < \infty$ on a measure space $(X, \mathrm{d}m)$ with finite measure, and if T_τ is a continuous one-parameter group of measure preserving transformations on X, then

$$\begin{aligned} F^*(x) &= \lim_{S\to\infty} \frac{1}{S} \int_0^S F(T_\tau(x))\, \mathrm{d}\tau &(25)\\ &= \lim_{\epsilon\to 0+} \epsilon \int_0^\infty \mathrm{e}^{-\epsilon\tau} F(T_\tau(x))\, \mathrm{d}\tau &(26) \end{aligned}$$

converges in the sense of L^p (and in particular almost everywhere). We apply this theorem to $X = \Sigma$, $\mathrm{d}m = \sqrt{\gamma}\, \mathrm{d}^{n-2}x$, $F = \alpha$, and $T_\tau = \hat{\phi}_\tau$, to conclude that there is an L^p function $\alpha^*(x)$ on Σ to which the limit in the lemma converges. We would like to prove that $\alpha^*(x)$

is constant. To prove this, we note that eq. (18) together with the facts that $\mathscr{L}_t\beta_a = 0$ and $t^a = n^a + s^a$ yields

$$D_b\alpha = -\frac{1}{2}\mathscr{L}_s\beta_b\,. \tag{27}$$

Now let

$$a(x,S) = \int_0^S \alpha(\hat{\phi}_\tau(x))\,\mathrm{d}\tau\,. \tag{28}$$

Then

$$D_b a(x,S) = -\frac{1}{2}\left\{\hat{\phi}^*_S\beta_b(x) - \beta_b(x)\right\}\,, \tag{29}$$

and thus

$$\begin{aligned} |a(x,S) - a(y,S)| &\leq C' \sup\{[D^b a D_b a(z,S)]^{1/2};\ z\in\Sigma\} \\ &\leq C' \sup\{[\beta^b\beta_b(z)]^{1/2};\ z\in\Sigma\} = C < \infty\,, \end{aligned} \tag{30}$$

where C, C' are constants independent of S and x, and where C is finite because Σ is compact. Consequently, $|a(x,S) - a(y,S)|$ is uniformly bounded in $S \geq 0$ and in $x, y \in \Sigma$. Thus, for all $x, y \in \Sigma$, we have

$$\lim_{S\to\infty}\frac{1}{S}|a(x,S) - a(y,S)| \leq \lim_{S\to\infty}\frac{C}{S} = 0\,. \tag{31}$$

Let $y \in \Sigma$ be such that $a(y,S)/S$ converges as $S \to \infty$. (As already noted above, existence of such a y is guaranteed by the von Neumann ergodic theorem.) The above equation then shows that, in fact, $a(x,S)/S$ must converge for all $x \in \Sigma$ as $S \to \infty$ and that, furthermore, the limit is independent of x, as we desired to show. Thus, $\alpha^*(x)$ is constant, and hence equal to its spatial average, κ. The estimate (30) also shows that the limit (24) is uniform in x. Similar estimates can easily be obtained for the norm with respect to γ_{ab} of $[D_{c_1}\cdots D_{c_k}a(x,S) - D_{c_1}\cdots D_{c_k}a(y,S)]$, for any k. These estimates show that x-derivatives of $a(x,S)/S$ converge to 0 uniformly in x. $\square$

We now are in a position to prove the existence of a positive function f on Σ satisfying the last equality in eq. (23) on Σ. Let

$$f(x) = \kappa\int_0^\infty p(x,\sigma)\,\mathrm{d}\sigma, \tag{32}$$

where $p(x,\sigma) > 0$ is the function on $\Sigma\times\mathbb{R}$ defined by

$$p(x,\sigma) = \exp\left(-\int_0^\sigma \alpha(\hat{\phi}_\tau(x))\,\mathrm{d}\tau\right)\,. \tag{33}$$

The function f is well defined for almost all x because $p(x,\sigma) < \mathrm{e}^{-\sigma(\kappa-\epsilon)}$ for any ϵ and sufficiently large σ, by Lemma 1. It also follows from the uniformity statement in Lemma 1 that f is smooth on Σ. By a direct calculation, using Lemma 1, we find that f satisfies

$$-\mathscr{L}_s f(x) + \alpha(x)f(x) = \kappa\,, \tag{34}$$

as we desired to show.

We now can deduce how to choose the desired new Gaussian null coordinates. The new coordinate $\tilde{u}$ must satisfy

$$\mathscr{L}_t\tilde{u} = 1\,, \tag{35}$$

as before. However, in view of eq. (21), it also must satisfy

$$\mathscr{L}_n \tilde{u} = n^a \nabla_a \tilde{u} = \frac{1}{f} \tilde{n}^a \nabla_a \tilde{u} = \frac{1}{f} \,. \tag{36}$$

Since $n^a = t^a - s^a$, we find that on Σ, $\tilde{u}$ must satisfy

$$1 - \mathscr{L}_s \tilde{u} = \frac{1}{f} \,. \tag{37}$$

Substituting from eq. (34), we obtain

$$\mathscr{L}_s \tilde{u} = 1 + \frac{1}{\kappa} (\mathscr{L}_s \ln f - \alpha) \,. \tag{38}$$

Thus, if our new Gaussian null coordinates exist, there must exist a smooth solution to this equation. That this is the case is proven in the following lemma.

Lemma 2 There exists a smooth solution h to the following differential equation on Σ:

$$\mathscr{L}_s h(x) = \alpha(x) - \kappa \,. \tag{39}$$

Proof: First note that the orbit average of any function of the form $\mathscr{L}_s h(x)$ where h is smooth must vanish, so there could not possibly exist a smooth solution to the above equation unless the average of α over any orbit is equal to κ. However, this was proven to hold in Lemma 1. In order to get a solution to the above equation, choose $\epsilon > 0$, and consider the regulated expression defined by

$$h_\epsilon(x) = - \int_0^\infty \mathrm{e}^{-\epsilon \tau} \left[\alpha(\hat{\phi}_\tau(x)) - \kappa \right] \mathrm{d}\tau \,. \tag{40}$$

Due to the exponential damping, this quantity is smooth, and satisfies the differential equation

$$\mathscr{L}_s h_\epsilon(x) = \alpha(x) - \kappa - \epsilon h_\epsilon(x) \,. \tag{41}$$

We would now like to take the limit as $\epsilon \to 0$ to get a solution to the desired equation. However, it is not possible to straightforwardly take the limit as $\epsilon \to 0$ of $h_\epsilon(x)$, for there is no reason why this should converge without using additional properties of α. In fact, we will not be able to show that the limit as $\epsilon \to 0$ of $h_\epsilon(x)$ exists, but we will nevertheless construct a smooth solution to eq. (39).

To proceed, we rewrite eq. (40) as

$$h_\epsilon(x) = - \int_0^\infty \mathrm{e}^{-\epsilon \tau} \hat{\phi}^*_\tau \alpha(x) \, \mathrm{d}\tau \, + \frac{\kappa}{\epsilon} \,, \tag{42}$$

where $\hat{\phi}^*_\tau$ denotes the pull-back map on tensor fields associated with $\hat{\phi}_\tau$. Taking the gradient of this equation and using eq. (27), we obtain

$$\mathrm{d}h_\epsilon(x) = - \int_0^\infty \mathrm{e}^{-\epsilon \tau} \hat{\phi}^*_\tau (\mathrm{d}\alpha)(x) \, \mathrm{d}\tau = \frac{1}{2} \int_0^\infty \mathrm{e}^{-\epsilon \tau} \hat{\phi}^*_\tau (\mathscr{L}_s \beta)(x) \, \mathrm{d}\tau \,, \tag{43}$$

where here and in the following we use differential forms notation and omit tensor indices. Since $\mathscr{L}_s$ clearly commutes with $\hat{\phi}^*_\tau$ and since $\mathscr{L}_s$ is just the derivative along the orbit over which we are integrating, we can integrate by parts to obtain

$$\mathrm{d}h_\epsilon(x) = -\frac{1}{2}\beta(x) + \frac{\epsilon}{2}\int_0^\infty \mathrm{e}^{-\epsilon\tau}\,\hat{\phi}^*_\tau\beta(x)\,\mathrm{d}\tau\,. \tag{44}$$

It follows from the von Neumann ergodic theorem[4] (see eq. (25)) that the limit

$$\lim_{\epsilon\to 0}\epsilon\int_0^\infty \mathrm{e}^{-\epsilon\tau}\,\hat{\phi}^*_\tau\beta(x)\,\mathrm{d}\tau = \beta^*(x)\,, \tag{45}$$

exists in the sense of $L^p(\Sigma)$. Furthermore, the limit in the sense of $L^p(\Sigma)$ also exists of all x-derivatives of the left side. Indeed, because $\hat{\phi}_\tau$ is an isometry commuting with the derivative operator D_a of the metric γ_{ab}, we have

$$D_{c_1}\cdots D_{c_k}\left(\epsilon\int_0^\infty \mathrm{e}^{-\epsilon\tau}\,\hat{\phi}^*_\tau\beta_a(x)\,\mathrm{d}\tau\right) = \epsilon\int_0^\infty \mathrm{e}^{-\epsilon\tau}\,\hat{\phi}^*_\tau D_{c_1}\cdots D_{c_k}\beta_a(x)\,\mathrm{d}\tau\,. \tag{46}$$

The expression on the right side converges in L^p, as $\epsilon\to 0$ by the von Neumann ergodic theorem, meaning that

$$\mathrm{d}h_\epsilon \to -\frac{1}{2}(\beta-\beta^*) \quad \text{in } W^{k,p}(\Sigma) \text{ as } \epsilon\to 0\,, \tag{47}$$

for all $k\geq 0$, $p\geq 1$, where $W^{k,p}(\Sigma)$ denotes the Sobolev space of order (k,p). By the Sobolev embedding theorem,

$$C^m(\Sigma)\hookleftarrow W^{k,p}(\Sigma) \quad \text{for } k > m + (n-2)/p\,, \tag{48}$$

where the embedding is continuous with respect to the sup norm on the all derivatives in the space C^m, i.e., $\sup_\Sigma|D^m\psi(x)|\leq \text{const.}\|\psi\|_{W^{k,p}}$ for all $\psi\in C^m$. Thus, convergence of the limit (45) actually occurs in the sup norms on C^m. Thus, in particular, $\beta^*\in C^\infty = \cap_{m\geq 0}C^m$.

Now pick an arbitrary $x_0\in\Sigma$, and define F_ϵ by

$$h_\epsilon(x) - h_\epsilon(x_0) = \int_{C(x)}\mathrm{d}h_\epsilon = F_\epsilon(x)\,, \tag{49}$$

where the integral is over any smooth path $C(x)$ connecting x_0 and x. This integral manifestly does not depend upon the choice of $C(x)$, independently of the topology of Σ. By what we have said above, the function F_ϵ is smooth, with a smooth limit

$$F(x) = \lim_{\epsilon\to 0}F_\epsilon(x) = -\frac{1}{2}\int_{C(x)}(\beta-\beta^*)\,, \tag{50}$$

[4]Here, the theorem is applied to the case of a tensor field T of type (k,l) on a compact Riemannian manifold Σ, rather than a scalar function, and where the measure preserving map is a smooth one-parameter family of isometries acting on T via the pull back. To prove this generalization, we note that a tensor field T of type (k,l) on a manifold Σ may be viewed as a function on the fiber bundle, B, of all tensors of type (l,k) over Σ that satisfies the additional property that this function is linear on each fiber. Equivalently, we may view T as a function, F, on the bundle, B', of unit norm tensors of type (l,k) that satisfies a corresponding linearity property. A Riemannian metric on Σ naturally gives rise to a Riemannian metric (and, in particular, a volume element) on B', and B' is compact provided that Σ is compact. Since the isometry flow on Σ naturally induces a volume preserving flow on B', we may apply the von Neumann ergodic theorem to F to obtain the orbit averaged function F^*. Since F^* will satisfy the appropriate linearity property on each fiber, we thereby obtain the desired orbit averaged tensor field T^*.

which is independent of the choice of $C(x)$. Furthermore, the convergence of F_ϵ and its derivatives to F and its derivatives is uniform. Now, by inspection, F_ϵ is a solution to the differential equation

$$\mathscr{L}_s F_\epsilon(x) = \alpha(x) - \kappa - \epsilon F_\epsilon(x) - \epsilon h_\epsilon(x_0)\,. \tag{51}$$

Furthermore, the limit

$$\begin{aligned}\lim_{\epsilon\to 0} \epsilon h_\epsilon(x_0) &= -\lim_{\epsilon\to 0} \epsilon \int_0^\infty \mathrm{e}^{-\tau\epsilon}\left[\alpha(\hat\phi_\tau(x_0)) - \kappa\right] \mathrm{d}\tau \\ &= \kappa - \lim_{S\to\infty} \frac{1}{S}\int_0^S \alpha(\hat\phi_\tau(x_0))\,\mathrm{d}\tau = 0 \end{aligned} \tag{52}$$

exists by the ergodic theorem, and vanishes by Lemma 1. Thus, the smooth, limiting quantity $F = \lim_{\epsilon\to 0} F_\epsilon$ satisfies the desired differential equation (39). □

We now define a new set of Gaussian null coordinates $(\tilde u, \tilde r, \tilde x^A)$ as follows. Define $\tilde u$ on Σ to be a smooth solution to eq. (38), whose existence is guaranteed by Lemma 2. Extend $\tilde u$ to H by eq. (35). It is not difficult to verify that $\tilde u$ is given explicitly by

$$\tilde u(x) = \int_0^{u(x)} \left[f\left(\pi\circ\phi_{-\tau}(x)\right)\right]^{-1} \mathrm{d}\tau + \frac{1}{\kappa}\ln f(\pi(x)) - \frac{1}{\kappa} h(\pi(x))\,, \tag{53}$$

where f and h are smooth solutions to eqs. (23) and (39), respectively, on Σ and

$$\pi : H \to \Sigma, \quad x \mapsto \pi(x) \tag{54}$$

is the map projecting any point x in H to the point $\pi(x)$ on the cross section Σ on the null generator through x. Let $\tilde\Sigma$ denote the surface $\tilde u = 0$ on H. Then our desired Gaussian null coordinates $(\tilde u, \tilde r, \tilde x^A)$ are the Gaussian null coordinates associated with $\tilde\Sigma$. The corresponding fields $\tilde\alpha, \tilde\beta_a, \tilde\gamma_{ab}$ satisfy all of the properties derived above for $\alpha, \beta_a, \gamma_{ab}$ and, in addition, satisfy the condition that $\tilde\alpha = \kappa$ is constant on H.

Now let $K^a = \tilde n^a$. We have previously shown that $\mathscr{L}_{\tilde n}\tilde\gamma_{ab} = 0$ on H, since this relation holds for any choice of Gaussian null coordinates. However, since our new coordinates have the property that $\tilde\alpha = \kappa$ is constant on H, we clearly have that $\mathscr{L}_{\tilde n}\tilde\alpha = 0$ on H. Furthermore, for our new coordinates, eq. (18) immediately yields $\mathscr{L}_{\tilde n}\tilde\beta_a = 0$ on H. Thus, we have proven that all of the relations in eq. (11) hold for $m = 0$.

We next prove that the equation $\mathscr{L}_{\tilde\ell}\mathscr{L}_{\tilde n}\tilde\gamma_{ab} = 0$ holds on H. Using what we already know about $\tilde\beta_a, \tilde\gamma_{ab}$ and taking the Lie-derivative $\mathscr{L}_{\tilde n}$ of the Einstein equation $R_{ab}(\partial/\partial\tilde x^A)^a(\partial/\partial\tilde x^B)^b = 0$ (see eq. (82) of Appendix A), we get

$$0 = \mathscr{L}_{\tilde n}\left[\mathscr{L}_{\tilde n}\mathscr{L}_{\tilde\ell}\tilde\gamma_{ab} + \kappa\,\mathscr{L}_{\tilde\ell}\tilde\gamma_{ab}\right]\,, \tag{55}$$

on H. Since $t^a = \tilde n^a + \tilde s^a$, with $\tilde s^a$ tangent to $\tilde\Sigma(\tilde u)$, and since all quantities appearing in eq. (55) are Lie derived by t^a, we may replace in this equation all Lie derivatives $\mathscr{L}_{\tilde n}$ by $-\mathscr{L}_{\tilde s}$. Hence, we obtain

$$0 = \mathscr{L}_{\tilde s}\left[\mathscr{L}_{\tilde s}\mathscr{L}_{\tilde\ell}\tilde\gamma_{ab} - \kappa\mathscr{L}_{\tilde\ell}\tilde\gamma_{ab}\right]\,, \tag{56}$$

on $\tilde\Sigma$. Now, write $L_{ab} = \mathscr{L}_{\tilde\ell}\tilde\gamma_{ab}$. We fix $x_0 \in \tilde\Sigma$ and view eq. (56) as an equation holding at x_0 for the pullback, $\hat\phi^*_\tau L_{ab}$, of L_{ab} to x_0, where $\hat\phi_\tau : \tilde\Sigma \to \tilde\Sigma$ now denotes the flow of $\tilde s^a$. Then eq. (56) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\mathrm{e}^{\kappa\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{e}^{-\kappa\tau}\hat\phi^*_\tau L_{ab})\right] = 0\,. \tag{57}$$

Integration of this equation yields

$$\mathrm{e}^{\kappa\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{e}^{-\kappa\tau}\hat{\phi}^*_\tau L_{ab}) = -\kappa C_{ab}\,, \tag{58}$$

where C_{ab} is a tensor at x_0 that is independent of τ. Integrating this equation (and absorbing constant factors into C_{ab}), we obtain

$$\hat{\phi}^*_\tau\, L_{ab} - \mathrm{e}^{\kappa\tau} L_{ab} = (1-\mathrm{e}^{\kappa\tau})C_{ab}\,. \tag{59}$$

However, since $\hat{\phi}_\tau$ is a Riemannian isometry, each orthonormal frame component of $\hat{\phi}^*_\tau\, L_{ab}$ at x_0 is uniformly bounded in τ by the Riemannian norm of L_{ab}, i.e., $\sup\{(L^{ab}L_{ab}(x))^{1/2};\ x \in \tilde{\Sigma}\}$. Consequently, the limit of eq. (59) as $\tau\to\infty$ immediately yields

$$L_{ab} = C_{ab} \tag{60}$$

from which it then immediately follows that

$$\hat{\phi}^*_\tau\, L_{ab} = L_{ab}\,. \tag{61}$$

Thus, we have $\mathscr{L}_{\tilde{s}}\mathscr{L}_{\tilde{\ell}}\tilde{\gamma}_{ab} = 0$, and therefore $\mathscr{L}_{\tilde{n}}\mathscr{L}_{\tilde{\ell}}\tilde{\gamma}_{ab} = \mathscr{L}_{\tilde{\ell}}\,\mathscr{L}_{\tilde{n}}\tilde{\gamma}_{ab} = 0$ on H, as we desired to show.

Thus, we now have shown that the first equation in (11) holds for $m = 0, 1$, and that the other equations hold for $m = 0$, for the tensor fields associated with the "tilde" Gaussian null coordinate system, and $K = \tilde{n}$. In order to prove that eq. (11) holds for all m, we proceed inductively. Let $M \geq 1$, and assume inductively that the first of equations (11) holds for all $m \leq M$, and that the remaining equations hold for all $m \leq M-1$. Our task is to prove that these statements then also hold when M is replaced by $M+1$. To show this, we apply the operator $(\mathscr{L}_{\tilde{\ell}})^{M-1}\mathscr{L}_{\tilde{n}}$ to the Einstein equation $R_{ab}\tilde{n}^a\tilde{\ell}^b = 0$ (see eq. (78)) and restrict to H. Using the inductive hypothesis, one sees that $(\mathscr{L}_{\tilde{\ell}})^M(\mathscr{L}_{\tilde{n}}\tilde{\alpha}) = 0$ on H, thus establishes the second equation in (11) for $m \leq M$. Next, we apply the operator $(\mathscr{L}_{\tilde{\ell}})^{M-1}\mathscr{L}_{\tilde{n}}$ to the Einstein equation $R_{ab}(\partial/\partial\tilde{x}^A)^a\tilde{\ell}^b = 0$ (see eq. (81)), and restrict to H. Using the inductive hypothesis, one sees that $(\mathscr{L}_{\tilde{\ell}})^M(\mathscr{L}_{\tilde{n}}\tilde{\beta}_a) = 0$ on H, thus establishes the third equation in (11) for $m \leq M$. Next, we apply the operator $(\mathscr{L}_{\tilde{\ell}})^M\mathscr{L}_{\tilde{n}}$ to the Einstein equation $R_{ab}(\partial/\partial\tilde{x}^A)^a(\partial/\partial\tilde{x}^B)^b = 0$ (see eq. (82)), and restrict to H. Using the inductive hypothesis and the above results $(\mathscr{L}_{\tilde{\ell}})^M(\mathscr{L}_{\tilde{n}}\tilde{\alpha}) = 0$ and $(\mathscr{L}_{\tilde{\ell}})^M(\mathscr{L}_{\tilde{n}}\tilde{\beta}_a) = 0$, one sees that the tensor field $L^{(M+1)}_{ab} \equiv (\mathscr{L}_{\tilde{\ell}})^{M+1}\tilde{\gamma}_{ab}$ satisfies a differential equation of the form

$$\mathscr{L}_{\tilde{n}}[\mathscr{L}_{\tilde{n}}L^{(M+1)}_{ab} + (M+1)\kappa\, L^{(M+1)}_{ab}] = 0 \tag{62}$$

on H. By the same argument as given above for L_{ab}, it follows that $\mathscr{L}_{\tilde{n}}\, L^{(M+1)}_{ab} = 0$. This establishes the first equation in (11) for $m \leq M+1$, and closes the induction loop.

Thus far, we have assumed only that the spacetime metric is smooth (C^∞). However, if we now assume that the spacetime is real analytic, and that H is an analytic submanifold, then it can be shown that the vector field K^a that we have defined above is, in fact, analytic. To see this, first note that if the cross section Σ of H is chosen to be analytic, then our Gaussian null coordinates are analytic, and, consequently, so is any quantity defined in terms of them, such as n^a and ℓ^a. Above, K^a was defined in terms of a certain special Gaussian normal coordinate system that was obtained from a geometrically special cross section. That cross section was obtained by a change (53) of the coordinate u. Thus, to show that K^a is analytic, we must show that this change of coordinates is analytic. By eq. (53), this will be the case provided that f and h are analytic. We prove this in Appendix C.

Since g_{ab} and K^a are analytic, so is $\mathscr{L}_K g_{ab}$. It follows immediately from the fact that this quantity and all of its derivatives vanish at any point of H that $\mathscr{L}_K g_{ab} = 0$ where defined, i.e., within the region where the Gaussian null coordinates $(\tilde{u}, \tilde{r}, \tilde{x}^A)$ are defined. This proves existence of a Killing field K^a in a neighborhood of the horizon. We may then extend K^a by analytic continuation. Now, analytic continuation need not, in general, give rise to a single-valued extension, so we cannot conclude that there exists a Killing field on the entire spacetime. However, by a theorem of Nomizu [31] (see also [4]), if the underlying domain is simply connected, then analytic continuation does give rise to a single-valued extension. By the topological censorship theorem [10, 11], the domain of outer communication has this property. Consequently, there exists a unique, single valued extension of K^a to the domain of outer communication, i.e., the exterior of the black hole (with respect to a given end of infinity). Thus, in the analytic case, we have proven the following theorem:

Theorem 1: Let (M, g_{ab}) be an analytic, asymptotically flat n-dimensional solution of the vacuum Einstein equations containing a black hole and possessing a Killing field t^a with complete orbits which are timelike near infinity. Assume that a connected component, H, of the event horizon of the black hole is analytic and is topologically $\mathbb{R} \times \Sigma$, with Σ compact and that $\kappa \neq 0$ (where κ is defined eq. (20) above). Then there exists a Killing field K^a, defined in a region that covers H and the entire domain of outer communication, such that K^a is normal to the horizon and K^a commutes with t^a.

The assumption of analyticity in this theorem can be partially removed in the following manner, using an argument similar to that given in [9]. Since $\kappa > 0$, the arguments of [34] show that the spacetime can be extended, if necessary, so that H is a proper subset of a regular bifurcate null surface H^* in some enlarged spacetime (M^*, g^*). We may then consider the characteristic initial value formulation for Einstein's equations [35, 29, 8] on this bifurcate null surface. Since the extended spacetime is smooth, the initial data induced on this bifurcate null surface should be regular. Since this initial data is invariant under the orbits of K^a, it follows that the solution to which this data gives rise will be invariant under a corresponding one-parameter group of diffeomorphisms in the domain of dependence, $D(H^*)$, of H^*. Thus, if one merely assumes that the spacetime is smooth, existence of a Killing field in $D(H^*)$ holds. However, since $D(H^*)$ lies inside the black hole, this argument does not show existence of a Killing field in the domain of outer communications. Interestingly, if one assumes that the spacetime is analytic—so that existence of a Killing field in the domain of outer communications follows from the above analytic continuation arguments—then this argument shows that the Killing field known to exist in the domain of outer communications also can be extended to all of $D(H^*)$.

3 Proof of existence of rotational Killing fields

We proved in the previous section that if the quantity κ defined by eq. (20) is non-vanishing, then there exists a vector field K^a in a neighborhood of H which is normal to H and is such that the equations (2) hold. As explained at the end of the previous section, in the analytic case, this implies the existence of a Killing field normal to the horizon in a region containing the horizon and the domain of outer communication. Since we are considering the case where t^a is not pointing along the null generators of H, the Killing field K^a is distinct from t^a. Hence, their difference $S^a \equiv \tilde{s}^a = t^a - K^a$ is also a nontrivial Killing field. There are two cases to consider:

1. The Killing field S^a has closed orbits, or
2. The Killing field S^a does not have closed orbits.

Only the first case can occur in 4-dimensions. In the first case, it follows immediately that the Killing field S^a corresponds to a rotation at infinity. The purpose of this section is to show that, in the second case, even though the orbits of S^a are not closed, there must exist $N \geq 2$ mutually commuting Killing fields, $\varphi^a_{(1)}, \dots, \varphi^a_{(N)}$, which possess closed orbits with period 2π and are such that

$$S^a = \Omega_1 \varphi^a_{(1)} + \cdots + \Omega_N \varphi^a_{(N)}\,, \tag{63}$$

for some constants Ω_i, all of whose ratios are irrational.

To simplify notation, throughout this section, we omit the "tildes" on all quantities, i.e., in this section $\ell^a, n^a, \Sigma, u, r, \alpha, \beta_a, \gamma_{ab}$ denote the quantities associated with our preferred Gaussian null coordinates. The Killing field S^a satisfies a number of properties that follow immediately from the construction of $K^a = n^a$ given in the previous section. First, since $\mathscr{L}_K r = 0 = \mathscr{L}_t r$ and since $\mathscr{L}_K \ell^a = 0 = \mathscr{L}_t \ell^a$, it follows that S^a also satisfies these properties, i.e., $\mathscr{L}_S r = 0$ and $\mathscr{L}_S \ell^a = 0$. Similarly, since $\mathscr{L}_K u = 1 = \mathscr{L}_t u$, we also have $\mathscr{L}_S u = 0$. In addition, since K^a commutes with t^a, so does S^a. Thus, S^a is tangent to the surfaces of constant (u, r), and commutes with ℓ^a and t^a. Finally, it follows immediately from eq. (11) and eqs. (13)-(15) that S^a also satisfies the analog of eq. (11).

To proceed, we focus attention now on the Riemannian manifold (Σ, γ_{ab}) and make arguments similar to those given in [21]. Let $\mathcal{G}$ denote the isometry group of (Σ, γ_{ab}). Then $\mathcal{G}$ is a compact Lie group. Let $\mathcal{H} \subset \mathcal{G}$ denote the one-parameter subgroup of $\mathcal{G}$ generated by the Killing field S^a on (Σ, γ_{ab}). Then the closure, $\overline{\mathcal{H}}$, of $\mathcal{H}$ is a closed subgroup of $\mathcal{G}$, and hence is a Lie subgroup. Since $\mathcal{H}$ is abelian, so is $\overline{\mathcal{H}}$, and since $\mathcal{G}$ is compact, it follows that $\overline{\mathcal{H}}$ is a torus. Let $N = \dim(\overline{\mathcal{H}})$. Since the N-dimensional torus can be written as the direct product of N factors of $U(1)$, it follows that the isometries in $\overline{\mathcal{H}}$ are generated by N commuting Killing vector fields, $\varphi^a_{(1)}, \dots, \varphi^a_{(N)}$, which possess closed orbits on Σ with period 2π. Since the isometry subgroup $\mathcal{H}$ generated by S^a is dense in $\overline{\mathcal{H}}$, it follows that, on Σ, S^a must be a linear combination of these Killing vector fields of the form (63).

Since, as we have noted above, S^a satisfies the analog of eq. (11), the diffeomorphisms on Σ corresponding to every element of $\mathcal{H}$ leave invariant each tensor field on Σ of the form

$$T_{(k)} = \begin{cases} \underbrace{\mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m+1 \text{ times}} \gamma_{ab} \\ \underbrace{\mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m \text{ times}} \beta_a \\ \underbrace{\mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m \text{ times}} \alpha\,, \end{cases} \tag{64}$$

for all $m \geq 0$. Since $\mathcal{H}$ is dense in $\overline{\mathcal{H}}$, each $T_{(k)}$ also must be invariant under the diffeomorphisms corresponding to the elements of $\overline{\mathcal{H}}$. Consequently, the Killing fields $\varphi^a_{(1)}, \dots, \varphi^a_{(N)}$ Lie derive all $T_{(k)}$ on Σ. We now extend each $\varphi^a_{(j)}$ to a vector field defined in an entire neighborhood of H as follows. First, we Lie-drag $\varphi^a_{(j)}$ from Σ to H via the vector field

$K^a = n^a$. Then we Lie-drag the resulting vector field defined on H off the horizon via the vector field ℓ^a. The vector field (denoted again by $\varphi^a_{(j)}$), which has now been defined in an entire neighborhood of H, satisfies the following properties throughout this neighborhood: (i) $\varphi^a_{(j)}$ commutes with both n^a and ℓ^a. (ii) $\varphi^a_{(j)}$ satisfies the analog of eq. (11). Property (ii) implies that in the analytic case, $\varphi^a_{(j)}$ is a Killing field of the spacetime metric. As was the case for K^a, we may then uniquely extend $\varphi^a_{(j)}$ as a Killing field to the entire domain of outer communication. That this extended Killing field (which we also denote by $\varphi^a_{(j)}$) must have closed orbits can be seen as follows: The orbits of $\varphi^a_{(j)}$ on Σ are closed with period 2π. Thus, if we consider the flow of $\varphi^a_{(j)}$ by parameter 2π, any point $x \in \Sigma$ will be mapped into itself, and vectors at x that are tangent to Σ also will get mapped into themselves. Furthermore, since $\varphi^a_{(j)}$ commutes with n^a and ℓ^a tangent vectors at x that are orthogonal to Σ also will get mapped into themselves. Consequently, the isometry on the spacetime corresponding to the action of $\varphi^a_{(j)}$ by parameter 2π maps point x into itself and maps each vector at x into itself. Consequently, this isometry is the identity map in any connected region where it is defined. Thus, we have shown:

Theorem 2: Let (M, g_{ab}) be an analytic, asymptotically flat n-dimensional solution of the vacuum Einstein equations containing a black hole and possessing a Killing field t^a with complete orbits which are timelike near infinity. Assume that a connected component, H, of the event horizon of the black hole is analytic and is topologically $\mathbb{R} \times \Sigma$, with Σ compact and that $\kappa \neq 0$ (where κ is defined eq. (20) above). If t^a is not tangent to the generators of H, then there exist mutually commuting Killing fields $\varphi^a_{(1)}, \ldots, \varphi^a_{(N)}$ (where $N \geq 1$) with closed orbits with period 2π which are defined in a region that covers H and the entire domain of outer communication. Each of these Killing fields commute with t^a, and t^a can be written as

$$t^a = K^a + \Omega_1 \varphi^a_{(1)} + \cdots + \Omega_N \varphi^a_{(N)} \,, \tag{65}$$

for some constants Ω_i, all of whose ratios are irrational, where K^a is the horizon Killing field whose existence is guaranteed by Theorem 1.

Theorem 2 shows that the null geodesic generators of the event horizon rotate rigidly with respect to infinity.

Remarks:

1) As in the case of K^a, in the non-analytic case the Killing fields $\varphi^a_{(1)}, \ldots, \varphi^a_{(N)}$ can be proven to exist in $D(H^*)$ (see the end of section 2).

2) If the orbits of S^a are closed on Σ—i.e., equivalently, if the orbits of t^a map each generator of H to itself after some period P—then the above argument shows that S^a itself is a Killing field with closed orbits. As previously noted in the introduction, in the case of 4-dimensional spacetimes, the orbits of S^a on Σ are always closed. However, the orbits of S^a on Σ need not be closed when $n > 4$. For example, on the round 3-sphere S^3, one can take an incommensurable linear combination of two commuting Killing fields with closed orbits to obtain a Killing field with non-closed orbits. This possibility is realized for S^a in suitably chosen 5-dimensional Myers-Perry black hole solutions [30]. Our theorem shows that if the orbits of S^a fail to be closed, then the spacetime must admit at least two linearly independent rotational Killing fields.

3) In 4-dimensions, if t^a is normal to the horizon, then Thm. 3.4 of [42] (applied to the vacuum or Einstein-Maxwell cases) shows that the exterior region must be static. The

proof of this result makes use of the fact that there exists a bifurcation surface (i.e., that we are in the non-degenerate case $\kappa \neq 0$), and that there exists a suitable foliation of the exterior region by maximal surfaces; the existence of such a foliation was proven in [5]. The arguments of [42] generalize straightforwardly to higher dimensions. Thus, the staticity of higher dimensional vacuum or Einstein-Maxwell stationary black holes with t^a normal to the horizon must hold provided that the arguments of [5] also generalize suitably to higher dimensions. [5] It should be noted that n-dimensional static vacuum (and Maxwell-dilaton) black hole spacetimes with a standard null infinity of topology $\mathscr{I} \cong S^{n-2} \times \mathbb{R}$ were shown to be essentially unique by Gibbons et al. [13], and are, in particular, spherically symmetric. However, static spacetimes with a non-trivial topology at infinity do not have to have any extra Killing fields [14]. [6]

4 Matter fields

The analysis of the foregoing sections can be generalized to include various matter sources, and we now illustrate this by discussing several examples. The Einstein equation with matter is

$$R_{ab} = T_{ab} - \frac{1}{n-2} g_{ab} T^c{}_c \,. \tag{66}$$

The simplest matter source is a cosmological constant $T_{ab} = -\Lambda g_{ab}$. The only significant change resulting from the presence of a cosmological constant is a change in the asymptotic properties of the spacetime. For the case of a negative cosmological constant, we can consider spacetimes that are asymptotically AdS rather than asymptotically flat. For asymptotically AdS spacetimes, $\mathscr{I}$ is no longer null, but is instead timelike. However, the only place where we used the character of $\mathscr{I}$ in our arguments was to conclude that the Killing field t^a is nowhere vanishing on H, and therefore generates a suitable foliation $\Sigma(u)$ of H by cross sections. This argument goes through without change in the asymptotically AdS case, as do all subsequent arguments in our proof. Thus, the rigidity theorem holds without modification in the case of a negative cosmological constant. For a positive cosmological constant, $\mathscr{I}$ would have a spacelike character and it is not clear precisely what should be assumed about the behavior of t^a near $\mathscr{I}$. Nevertheless, our results apply straightforwardly to any "horizon" that is the boundary of the past of any complete, timelike orbit of t^a. In this sense, our rigidity theorem holds for both black hole event horizons and cosmological horizons.

For Maxwell fields the field equations and stress tensor are

$$\nabla^a F_{ab} = 0 \,, \quad \nabla_{[a} F_{bc]} = 0 \,, \quad T_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F^{cd} F_{cd} \,. \tag{67}$$

We assume that both metric and Maxwell tensor are invariant under t^a, i.e., $\mathscr{L}_t g_{ab} = 0 = \mathscr{L}_t F_{ab}$. In parallel with the vacuum case, we wish to show that there exists a vector field K^a tangent to the generators of the horizon satisfying

$$\underbrace{\mathscr{L}_\ell \mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m \text{ times}} (\mathscr{L}_K g_{ab}) = 0 \,, \quad \underbrace{\mathscr{L}_\ell \mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m \text{ times}} (\mathscr{L}_K F_{ab}) = 0 \,, \quad m = 0, 1, 2, \ldots \,, \tag{68}$$

[5] It has recently been shown in [41] that this is indeed the case.

[6] See also [37, 38, 39] for uniqueness results of higher dimensional static black holes and [28, 40] for uniqueness results of some restricted class of 5-dimensional stationary black holes.

on H. To analyze these equations we introduce a Gaussian null coordinate system as above, and correspondingly decompose the field strength tensor as

$$F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = S\,\mathrm{d}u \wedge \mathrm{d}r + V_A\,\mathrm{d}u \wedge \mathrm{d}x^A + W_A\,\mathrm{d}r \wedge \mathrm{d}x^A + U_{AB}\,\mathrm{d}x^A \wedge \mathrm{d}x^B\,. \tag{69}$$

We write

$$V_a = V_A(\mathrm{d}x^A)_a\,, \quad W_a = W_A(\mathrm{d}x^A)_a\,, \quad U_{ab} = U_{AB}(\mathrm{d}x^A)_a(\mathrm{d}x^B)_b\,. \tag{70}$$

It follows from $\mathscr{L}_t F_{ab} = 0$ that $\mathscr{L}_t S = \mathscr{L}_t V_a = \mathscr{L}_t W_a = \mathscr{L}_t U_{ab} = 0$. As in the vacuum case, we take the candidate Killing field K^a to be n^a, where n^a is the vector field associated with a suitable Gaussian null coordinate system to be determined. Equations (68) are then equivalent to eqs. (11) together with

$$\begin{aligned}
\underbrace{\mathscr{L}_\ell \mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m \text{ times}}(\mathscr{L}_n S) &= 0\,,\\
\underbrace{\mathscr{L}_\ell \mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m \text{ times}}(\mathscr{L}_n V_a) &= 0\,,\\
\underbrace{\mathscr{L}_\ell \mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m \text{ times}}(\mathscr{L}_n W_a) &= 0\,,\\
\underbrace{\mathscr{L}_\ell \mathscr{L}_\ell \cdots \mathscr{L}_\ell}_{m \text{ times}}(\mathscr{L}_n U_{ab}) &= 0\,.
\end{aligned} \tag{71}$$

The Maxwell equations, the Bianchi identities, and the stress tensor are presented in Appendix B. The Raychaudhuri equation now gives $\mathscr{L}_n \gamma_{ab} = 0 = V_a$ on H. Then, the Bianchi identity (87) yields $\mathscr{L}_n U_{ab} = 0$. The Maxwell equation (84) yields $\mathscr{L}_n S = 0$ on H. Furthermore, we have $T_{ab} n^a (\partial/\partial x^A)^b = 0$ (see eq. (92)), from which it follows in view of Einstein's equation that also $R_{ab} n^a (\partial/\partial x^A)^b = 0$ on H. We may now argue in precisely the same way as in the vacuum case that, by a suitable choice of Gaussian null coordinates, we can achieve that $\mathscr{L}_n \alpha = 0 = \mathscr{L}_n \beta_a$ on H. It then follows from eqs. (95) and (96) that $\mathscr{L}_n [T_{ab} (\partial/\partial x^A)^a (\partial/\partial x^B)^b] = 0$ and $\mathscr{L}_n T^a{}_a = 0$ on H, which in view of Einstein's equation means that $\mathscr{L}_n [R_{ab} (\partial/\partial x^A)^a (\partial/\partial x^B)^b] = 0$ on H. This may in turn be used to argue, precisely as in the vacuum case, that $\mathscr{L}_\ell \mathscr{L}_n \gamma_{ab} = 0$ on H. Taking a Lie derivative $\mathscr{L}_n$ of the Maxwell equation (85) and the Bianchi identity (86) then leads to the equation

$$\mathscr{L}_n \left[\mathscr{L}_n W_a + \kappa\, W_a\right] = 0\,, \tag{72}$$

on H. Since W_a is Lie derived by t^a and since $t^a = n^a + s^a$ as in the vacuum case, this equation may alternatively be written as

$$\mathscr{L}_s \left[\mathscr{L}_s W_a - \kappa\, W_a\right] = 0\,. \tag{73}$$

Integration gives

$$\hat{\phi}^*_\tau W_a - \mathrm{e}^{\kappa\tau} W_a = (1 - \mathrm{e}^{\kappa\tau}) C_a\,, \tag{74}$$

where C_a is a 1-form field on Σ independent of τ, and where $\hat{\phi}_\tau$ is again the flow of s^a. The same type of argument following eq. (59) then implies that $\mathscr{L}_n W_a = 0$ on H. We have thus shown that all eqs. (71) and (11) for $m = 0$ and the first equation in (11) for $m = 1$ are satisfied on the horizon. The remainder of the argument closely parallels the vacuum case.

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A Ricci tensor in Gaussian null coordinates

In this Appendix, we provide expressions for the Ricci tensor in a Gaussian null coordinate system. As derived in section 2, in Gaussian null coordinates, the metric takes the form

$$g_{ab} = 2\left(\nabla_{(a}r - r\alpha\nabla_{(a}u - r\beta_{(a)}\right)\nabla_{b)}u + \gamma_{ab}\,, \tag{75}$$

where the tensor fields β_a and γ_{ab} are orthogonal to n^a and ℓ^a. The horizon, H, corresponds to the surface $r = 0$. We previously noted that $\gamma^a{}_b$ is the orthogonal projector onto the subspace of the tangent space orthogonal to n^a and ℓ^a, and that when $r\beta_a \neq 0$, it differs from the orthogonal projector, $q^a{}_b$, onto the surfaces $\Sigma(u,r)$. It is worth noting that in terms of the Gaussian null coordinate components of γ_{ab}, we have $q^{ab} = (\gamma^{-1})^{AB}(\partial/\partial x^A)^a(\partial/\partial x^B)^b$. It also is convenient to introduce the non-orthogonal projector $p^a{}_b$, uniquely defined by the conditions that $p^a{}_b n^b = p^a{}_b \ell^b = 0$ and that $p^a{}_b$ be the identity map on vectors that are tangent to $\Sigma(u,r)$. The relationship between $p^a{}_b$ and $\gamma^a{}_b$ is given by

$$p^a{}_b = -r\ell^a\beta_b + \gamma^a{}_b\,. \tag{76}$$

In terms of Gaussian null coordinates, we have $p^a{}_b = (\partial/\partial x^A)^a(\mathrm{d}x^A)_b$, from which it is easily seen that $\mathscr{L}_n p^a{}_b = 0 = \mathscr{L}_\ell p^a{}_b$. It also is easily seen that $q^{ac}\gamma_{cb} = p^a{}_b$ and that $p^a{}_b q^b{}_c = q^a{}_c$.

We define the derivative operator D_c acting on a tensor field $T^{a_1\ldots a_r}{}_{b_1\ldots b_s}$ by the following prescription. First, we project the indices of the tensor field by $q^a{}_b$, then we apply the covariant derivative ∇_c, and we then again project all indices using $q^a{}_b$. For tensor fields intrinsic to Σ, this corresponds to the derivative operator associated with the metric q_{ab}. We denote the Riemann and Ricci tensors associated with q_{ab} as $\mathcal{R}_{abc}{}^d$ and $\mathcal{R}_{ab}$.

The Ricci tensor of g_{ab} can then be written in the following form:

$$
\begin{aligned}
n^a n^b R_{ab} \;=\; & -\frac{1}{2} q^{ab} \mathscr{L}_n \mathscr{L}_n \gamma_{ab} + \frac{1}{4} q^{ca} q^{db} (\mathscr{L}_n \gamma_{ab}) \mathscr{L}_n \gamma_{cd} + \frac{1}{2} \alpha\, q^{ab} \mathscr{L}_n \gamma_{ab} \\
+ & \frac{r}{2} \cdot \Bigg[4\alpha \mathscr{L}_\ell \mathscr{L}_\ell \alpha + 8\alpha \mathscr{L}_\ell \alpha + (\mathscr{L}_\ell \alpha) q^{ab} \mathscr{L}_n \gamma_{ab} \\
& + q^{ab} \mathscr{L}_\ell \gamma_{ab} \cdot \Big\{ - \mathscr{L}_n \alpha - r q^{cd} \beta_c \mathscr{L}_n \beta_d \\
& \qquad + (r q^{cd} \beta_c \beta_d + 2\alpha) \mathscr{L}_\ell (r\alpha) + r q^{cd} \beta_c D_d \alpha \Big\} \\
& + 2 q^{ab} D_a \left\{ \beta_b \mathscr{L}_\ell (r\alpha) + D_b \alpha - \mathscr{L}_n \beta_b \right\} \\
& + q^{bc} \mathscr{L}_\ell (r\beta_c) \cdot \Big\{ (r q^{ef} \beta_e \beta_f + 2\alpha) \mathscr{L}_\ell (r\beta_b) \\
& \qquad - 4 D_b \alpha + 2 \mathscr{L}_n \beta_b + 4 r q^{ae} \beta_e D_{[a} \beta_{b]} \Big\} \\
& + 2 (\mathscr{L}_\ell \alpha) \mathscr{L}_\ell (r^2 q^{ab} \beta_a \beta_b) + 4 r q^{ab} \beta_a \beta_b \mathscr{L}_\ell \alpha + 2 r q^{ab} \beta_a \beta_b \mathscr{L}_\ell \mathscr{L}_\ell \alpha \\
& + 2 q^{ab} \beta_a \mathscr{L}_\ell (r\beta_b) \cdot \Big\{ 2 \mathscr{L}_\ell (r\alpha) - \frac{1}{2} r q^{cd} \beta_c \mathscr{L}_\ell (r\beta_d) \Big\} \\
& + 2 r^{-1} \mathscr{L}_\ell \Big\{ r^2 q^{ab} \beta_a (D_b \alpha - \mathscr{L}_n \beta_b) \Big\} + 2 r^{-1} \alpha \mathscr{L}_\ell (r^2 q^{ab} \beta_a \beta_b) \Bigg], \quad (77)
\end{aligned}
$$

$$
\begin{aligned}
n^a \ell^b R_{ab} \;=\; & -2 \mathscr{L}_\ell \alpha + \frac{1}{4} q^{ca} q^{db} (\mathscr{L}_n \gamma_{cd}) \mathscr{L}_\ell \gamma_{ab} - \frac{1}{2} q^{ab} \mathscr{L}_\ell \mathscr{L}_n \gamma_{ab} - \frac{1}{2} \alpha\, q^{ab} \mathscr{L}_\ell \gamma_{ab} - \frac{1}{2} q^{ab} \beta_a \beta_b \\
+ & \frac{r}{2} \cdot \Bigg[- 2 \mathscr{L}_\ell \mathscr{L}_\ell \alpha - \frac{1}{2} q^{ab} \mathscr{L}_\ell \gamma_{ab} \cdot \Big\{ 2 \mathscr{L}_\ell \alpha + q^{cd} \beta_c \mathscr{L}_\ell (r\beta_d) \Big\} \\
& - q^{ab} \beta_a \mathscr{L}_\ell \beta_b - \mathscr{L}_\ell \{ q^{ab} \beta_a \mathscr{L}_\ell (r\beta_b) \} - q^{ab} D_a (\mathscr{L}_\ell \beta_b) \Bigg], \quad (78)
\end{aligned}
$$

$$
\begin{aligned}
n^b p^c{}_a R_{bc} &= -p^b{}_a D_b\alpha + \frac{1}{2}\mathscr{L}_n\beta_a + \frac{1}{4}\beta_a q^{bc}\mathscr{L}_n\gamma_{bc} - p^d{}_{[a}p^e{}_{b]}D_d(q^{bc}\mathscr{L}_n\gamma_{ce}) \\
&+ \frac{r}{2}\cdot\Bigg[\frac{1}{2}(q^{bc}\mathscr{L}_n\gamma_{bc})\mathscr{L}_\ell\beta_a + \mathscr{L}_n\mathscr{L}_\ell\beta_a + 2\alpha\mathscr{L}_\ell\beta_a \\
&\quad +\mathscr{L}_\ell(r\beta_a)\cdot\Big\{r^{-1}\mathscr{L}_\ell(r^2q^{bc}\beta_b\beta_c) + 2\mathscr{L}_\ell\alpha\Big\} \\
&\quad -2p^b{}_a D_b(\mathscr{L}_\ell\alpha) + \mathscr{L}_\ell(q^{bc}\beta_b\mathscr{L}_n\gamma_{ca}) - 2r^{-1}\mathscr{L}_\ell\left(r^2q^{cd}\beta_c p^b{}_a D_{[b}\beta_{d]}\right) \\
&\quad -\frac{1}{2}q^{bc}\mathscr{L}_\ell\gamma_{bc}\cdot\Big\{-(rq^{ef}\beta_e\beta_f + 2\alpha)\mathscr{L}_\ell(r\beta_a) \\
&\qquad\qquad +2p^d{}_a D_d\alpha - q^{bc}\beta_b\mathscr{L}_n\gamma_{ca} + 2rq^{ef}\beta_e p^d{}_a D_{[d}\beta_{f]}\Big\} \\
&\quad -2\mathscr{L}_\ell(\alpha\beta_a) - 2r(\mathscr{L}_\ell\alpha)\mathscr{L}_\ell\beta_a + p^d{}_a D_b\Big\{q^{bc}\beta_c\mathscr{L}_\ell(r\beta_d)\Big\} \\
&\quad -2p^b{}_a q^{cd}D_d D_{[b}\beta_{c]} - q^{bc}(\mathscr{L}_\ell\beta_b)\mathscr{L}_n\gamma_{ca} \\
&\quad -q^{bc}\mathscr{L}_\ell(r\beta_b)\cdot\Big\{(rq^{ef}\beta_e\beta_f + 2\alpha)\mathscr{L}_\ell\gamma_{ca} + p^d{}_a D_c\beta_d \\
&\qquad\qquad +\beta_c\mathscr{L}_\ell(r\beta_a) - rq^{ef}\beta_c\beta_f\mathscr{L}_\ell\gamma_{ea}\Big\} \\
&\quad +q^{bc}(\mathscr{L}_\ell\gamma_{ca})\cdot\Big\{2\beta_b\mathscr{L}_\ell(r\alpha) + 2D_b\alpha - \mathscr{L}_n\beta_b + 2rq^{de}\beta_e D_{[b}\beta_{d]}\Big\}\Bigg], \quad (79)
\end{aligned}
$$

$$
\ell^a\ell^b R_{ab} = -\frac{1}{2}q^{ab}\mathscr{L}_\ell\mathscr{L}_\ell\gamma_{ab} + \frac{1}{4}q^{ca}q^{db}(\mathscr{L}_\ell\gamma_{ab})\mathscr{L}_\ell\gamma_{cd}\,, \quad (80)
$$

$$
\begin{aligned}
\ell^b p^c{}_a R_{bc} &= -\frac{1}{4}\beta_a q^{bc}\mathscr{L}_\ell\gamma_{bc} - \mathscr{L}_\ell\beta_a + \frac{1}{2}q^{bc}\beta_c\mathscr{L}_\ell\gamma_{ab} - p^d{}_{[a}p^e{}_{b]}D_d\left(q^{bc}\mathscr{L}_\ell\gamma_{ce}\right) \\
&+ \frac{r}{2}\cdot\Bigg[-\mathscr{L}_\ell\mathscr{L}_\ell\beta_a + \mathscr{L}_\ell\left(q^{bc}\beta_c\mathscr{L}_\ell\gamma_{ab}\right) \\
&\qquad +\frac{1}{2}(q^{cd}\mathscr{L}_\ell\gamma_{cd})\left(-\mathscr{L}_\ell\beta_a + q^{be}\beta_e\mathscr{L}_\ell\gamma_{ab}\right)\Bigg], \quad (81)
\end{aligned}
$$

$$
\begin{aligned}
p^c{}_a p^d{}_b R_{cd} &= -\mathscr{L}_\ell \mathscr{L}_n \gamma_{ab} - \alpha \mathscr{L}_\ell \gamma_{ab} + p^c{}_a p^d{}_b \mathcal{R}_{cd} - p^c{}_{(a} p^d{}_{b)} D_c \beta_d - \frac{1}{2}\beta_a\beta_b \\
&+ q^{cd}\left(\mathscr{L}_\ell \gamma_{d(a}\right) \mathscr{L}_n \gamma_{b)c} - \frac{1}{4}\left\{(q^{cd}\mathscr{L}_n\gamma_{cd})\mathscr{L}_\ell\gamma_{ab} + (q^{cd}\mathscr{L}_\ell\gamma_{cd})\mathscr{L}_n\gamma_{ab}\right\} \\
&+ \frac{r}{2}\cdot\Bigg[-2\alpha\mathscr{L}_\ell\mathscr{L}_\ell\gamma_{ab} - p^e{}_a p^f{}_b D_c(q^{cd}\beta_d\mathscr{L}_\ell\gamma_{ef}) \\
&\quad -\frac{1}{2}(q^{cd}\mathscr{L}_\ell\gamma_{cd})\left\{(rq^{ef}\beta_e\beta_f + 2\alpha)\mathscr{L}_\ell\gamma_{ab} + 2p^e{}_{(a}p^f{}_{b)}D_e\beta_f\right\} \\
&\quad -2(\mathscr{L}_\ell\alpha)\mathscr{L}_\ell\gamma_{ab} - r^{-1}\{\mathscr{L}_\ell(r^2 q^{ef}\beta_e\beta_f)\}\mathscr{L}_\ell\gamma_{ab} \\
&\quad -rq^{ef}\beta_e\beta_f\mathscr{L}_\ell\mathscr{L}_\ell\gamma_{ab} - 2\mathscr{L}_\ell\{p^c{}_{(a}p^d{}_{b)}D_c\beta_d\} \\
&\quad -2\beta_{(a}\mathscr{L}_\ell\beta_{b)} - r(\mathscr{L}_\ell\beta_a)\mathscr{L}_\ell\beta_b - rq^{ce}q^{df}\beta_c\beta_d(\mathscr{L}_\ell\gamma_{ae})\mathscr{L}_\ell\gamma_{bf} \\
&\quad +2q^{cd}\beta_d\left\{\mathscr{L}_\ell(r\beta_{(a})\right\}\mathscr{L}_\ell\gamma_{b)c} + 2p^e{}_{(a}p^f{}_{b)}q^{cd}\left(D_d\beta_e\right)\mathscr{L}_\ell\gamma_{fc} \\
&\quad +q^{cd}(rq^{ef}\beta_e\beta_f + 2\alpha)(\mathscr{L}_\ell\gamma_{ca})\mathscr{L}_\ell\gamma_{db}\Bigg]. \qquad (82)
\end{aligned}
$$

B Maxwell equations in Gaussian null coordinates

With the notation introduced in Appendix A and the definitions (69) and (70), the Maxwell equations, $\nabla_a F^{ab} = 0$, are equivalent to the following equations.

$$
\begin{aligned}
0 &= \mathscr{L}_\ell S + \frac{1}{2}Sq^{ab}\mathscr{L}_\ell\gamma_{ab} - q^{ab}\beta_a W_b - q^{ab}D_a W_b \\
&- \frac{r}{2}\cdot\Big[\,2\mathscr{L}_\ell(q^{ab}\beta_a W_b) + q^{ab}q^{cd}\beta_a W_b\mathscr{L}_\ell\gamma_{cd}\Big], \qquad (83) \\
0 &= \mathscr{L}_n S + \frac{1}{2}Sq^{ab}\mathscr{L}_n\gamma_{ab} + q^{ab}D_a V_b \\
&- \frac{r}{2}\cdot\Big[\,2\mathscr{L}_n(q^{ab}\beta_a W_b) + q^{ab}q^{cd}\beta_c W_d\mathscr{L}_n\gamma_{ab} \\
&\quad -2q^{ab}D_a\left\{SV_b + 2\alpha W_b - q^{cd}\beta_d U_{bc}\right\} - 4rq^{ab}q^{cd}D_a(\beta_c\beta_{[d}W_{b]})\Big], \qquad (84) \\
0 &= \frac{1}{2}q^{ab}q^{cd}\left(W_b\mathscr{L}_n\gamma_{cd} + V_b\mathscr{L}_\ell\gamma_{cd}\right) + \mathscr{L}_n(q^{ab}W_b) + \mathscr{L}_\ell(q^{ab}V_b) \\
&+ q^{ab}\left\{S\beta_b + 2\alpha W_b - q^{cd}\beta_d U_{bc}\right\} \\
&- \frac{r}{2}\cdot\Big[-2\mathscr{L}_\ell\left\{q^{ab}\left(S\beta_b + 2\alpha W_b - q^{cd}\beta_d U_{bc}\right)\right\} \\
&\quad -8q^{ab}q^{cd}\beta_c\beta_{[d}W_{b]} - 4\mathscr{L}_\ell\left(q^{ab}q^{cd}\beta_c\beta_{[d}W_{b]}\right) \\
&\quad -q^{cd}(\mathscr{L}_\ell\gamma_{cd})\cdot q^{ab}\left\{S\beta_b + q^{ef}\beta_e U_{fb} + 2\alpha W_b - 2rq^{ef}\beta_e\beta_{[b}W_{f]}\right\}\Big]. \qquad (85)
\end{aligned}
$$

The Bianchi identities, $\nabla_{[a}F_{bc]} = 0$, are given by

$$
\begin{aligned}
\mathscr{L}_n W_a - \mathscr{L}_\ell V_a + p^c{}_a D_c S &= 0\,, \qquad (86) \\
\mathscr{L}_n U_{ab} - 2p^c{}_{[a}p^d{}_{b]}D_c V_d &= 0\,, \qquad (87) \\
\mathscr{L}_\ell U_{ab} - 2p^c{}_{[a}p^d{}_{b]}D_c W_d &= 0\,, \qquad (88) \\
p^d{}_{[a}p^e{}_b p^f{}_{c]}D_d U_{ef} &= 0\,. \qquad (89)
\end{aligned}
$$

The stress tensor, $T_{ab} = F_{ac}F_b{}^c - (1/4)g_{ab}F^{cd}F_{cd}$, is given by

$$\begin{aligned}
n^a n^b T_{ab} &= q^{ab}V_aV_b + r\cdot\left[2q^{ab}\beta_aV_bS + (rq^{ab}\beta_a\beta_b + 2\alpha)S^2 + \frac{1}{2}\alpha F^{cd}F_{cd}\right], && (90)\\
n^a\ell^bT_{ab} &= -\frac{1}{2}S^2 - \frac{1}{4}q^{ac}q^{bd}U_{ab}U_{cd}\\
&+ r\cdot\left[\alpha q^{cd}W_cW_d + q^{bc}\beta_c\left(2W_bS + q^{de}W_eU_{bd}\right)\right.\\
&\qquad\left.+\frac{r}{2}\left(q^{ab}q^{cd} - q^{ac}q^{bd}\right)\beta_a\beta_bW_cW_d\right], && (91)\\
n^bp^c{}_aT_{bc} &= -SV_a + q^{bc}U_{ab}V_c\\
&+ r\cdot\left[q^{bc}\beta_c\left(-W_aV_b + U_{ab}S\right) - \left(rq^{bc}\beta_b\beta_c + 2\alpha\right)W_aS + \frac{1}{4}\beta_aF^{cd}F_{cd}\right], && (92)\\
\ell^a\ell^bT_{ab} &= q^{ab}W_aW_b\,, && (93)\\
\ell^bp^c{}_aT_{bc} &= SW_a + q^{bc}U_{ab}W_c - rq^{bc}\beta_cW_aW_b\,, && (94)\\
p^c{}_ap^d{}_bT_{cd} &= \frac{1}{2}\gamma_{ab}S^2 + 2V_{(a}W_{b)} - \gamma_{ab}q^{cd}V_cW_d + q^{cd}\left\{U_{ac}U_{bd} - \frac{1}{4}\gamma_{ab}q^{ef}U_{ce}U_{df}\right\}\\
&- r\cdot\gamma_{ab}\cdot\left[\alpha q^{bc}W_bW_c + q^{bc}\beta_c\left(W_bS + q^{de}W_eU_{bd}\right)\right.\\
&\qquad\left.+\frac{r}{2}\left(q^{cd}q^{ef} - q^{ce}q^{df}\right)\beta_c\beta_dW_eW_f\right], && (95)\\
T^c{}_c &= (n-4)\left\{\frac{1}{2}S^2 - q^{ab}V_aW_b - \frac{1}{4}q^{ab}q^{cd}U_{ac}U_{bd}\right\}\\
&- \frac{(n-4)}{2}\cdot r\cdot\left[2\alpha q^{ab}W_aW_b + 2q^{ab}\beta_a\left(W_bS + q^{cd}W_dU_{bc}\right)\right.\\
&\qquad\left.+r\left(q^{ab}q^{cd} - q^{ac}q^{bd}\right)\beta_a\beta_bW_cW_d\right], && (96)
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{4}F^{cd}F_{cd} &= q^{ab}V_aW_b - \frac{1}{2}S^2 + \frac{1}{4}q^{ac}q^{bd}U_{ab}U_{cd}\\
&+ r\cdot\left[\alpha q^{cd}W_cW_d + q^{bc}\beta_c\left(W_bS + q^{de}W_eU_{bd}\right)\right.\\
&\qquad\left.+\frac{r}{2}\left(q^{ab}q^{cd} - q^{ac}q^{bd}\right)\beta_a\beta_bW_cW_d\right]. && (97)
\end{aligned}$$

C Analyticity of f and h

In this Appendix, we prove the following lemma, which establishes that if the spacetime (M, g_{ab}) and horizon H are analytic, then the candidate Killing field K^a also is analytic.

Lemma 3: If the spacetime as well as Σ and H are analytic, then the functions $f, h : \Sigma \to \mathbb{R}$ given by eqs. (32) and (39) are real analytic.

Proof: Consider first the function $p(x, \sigma)$ on $\Sigma \times \mathbb{R}$ defined above in eq. (33). Let $x_0 \in \Sigma$ be fixed, and choose Riemannian normal coordinates $y^1, \dots, y^{n-2}$ around x_0, so that the coordinate components $\gamma_{AB}(x_0)$ are given by δ_{AB}. If $\alpha = (\alpha_1, \dots, \alpha_{n-2}) \in \mathbb{N}_0^{n-2}$ is a multi-index, we set $|\alpha| = \sum \alpha_i$, and $\alpha! = \prod \alpha_i!$, as well as

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{(\partial y^1)^{\alpha_1} \cdots (\partial y^{n-2})^{\alpha_{n-2}}} \,. \tag{98}$$

We will show that, for y in a sufficiently small ball around x_0, and for sufficiently large σ we have the following estimate:

$$|\partial^\alpha p(y, \sigma)| \le \alpha! C R^{-|\alpha|} \mathrm{e}^{-\sigma\kappa/2} \,, \tag{99}$$

with C and R being some constants independent of α. This implies that $f(y)$ has a convergent power series representation near x_0. Using Einstein's equation as in the proof of Lemma 1, we have

$$\frac{\partial}{\partial y^A} p(y, \sigma) = [\beta_A(y) - (\hat{\phi}_\sigma^* \beta)_A(y)] \, p(y, \sigma) \,. \tag{100}$$

Now we complexify Σ and consider a complex multi-disk around x_0 of radius R. Then, using the multi-dimensional version of the Cauchy inequalities, we furthermore have the estimate

$$|\partial^\alpha \beta_A(y)| \le \alpha! C R^{-|\alpha|} \,, \tag{101}$$

where C is now taken as the supremum of β_A in the complex multi-disk around x_0. Furthermore, because Σ is compact, and because $\hat{\phi}_\sigma$ is an isometry, we can find the same type of estimate in also for $\partial^\alpha (\hat{\phi}_\sigma^* \beta)_A(y)$ uniformly in σ and y in a ball around x_0. Finally, from Lemma 1, we have the estimate

$$|p(y, \sigma)| \le \mathrm{e}^{-\sigma(\kappa - \epsilon)} \,, \tag{102}$$

for arbitrary small $\epsilon > 0$ for any y in a ball around x_0, and for sufficiently large σ. Applying now further derivatives to eq. (100) and using the above estimates, we obtain the estimate (99).

In order to prove the analyticity of h, we must look at the explicit construction of that function given in the proof of Lemma 2. That construction shows that h will be analytic, if we can show that the vector field $\beta^*(x)$ defined by the integral (45) is analytic. This follows from the fact that the Taylor coefficient of the integrand $\partial^\alpha (\hat{\phi}_\tau^* \beta)_A(y)$ satisfies an estimate of the form (101) uniformly in τ and y in a ball around x_0. □

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