

# Asymptotic generators of fermionic charges and boundary conditions preserving supersymmetry

Stefan Hollands\*

Institut für Theoretische Physik

Georg-August Universität, D-37077 Göttingen, Germany

Donald Marolf†

Department of Physics, UCSB, Santa Barbara, CA 93106

December 20, 2006

## Abstract

We use a covariant phase space formalism to give a general prescription for defining Hamiltonian generators of bosonic and fermionic symmetries in diffeomorphism invariant theories, such as supergravities. A simple and general criterion is derived for a choice of boundary condition to lead to conserved generators of the symmetries on the phase space. In particular, this provides a criterion for the preservation of supersymmetries. For bosonic symmetries corresponding to diffeomorphisms, our prescription coincides with the method of Wald et al.

We then illustrate these methods in the case of certain supergravity theories in  $d = 4$ . In minimal AdS supergravity, boundary conditions such that the supercharges exist as Hamiltonian generators of supersymmetry transformations are unique within the usual framework in which the boundary metric is fixed. In extended  $\mathcal{N} = 4$  AdS supergravity, or more generally in the presence of chiral matter superfields, we find that there exist many boundary conditions preserving  $\mathcal{N} = 1$  supersymmetry for which corresponding generators exist. These choices are shown to correspond to a choice of certain arbitrary boundary “superpotentials,” for suitably defined “boundary superfields.” We also derive corresponding formulae for the conserved bosonic charges, such as energy, in those theories, and we argue that energy is always positive, for any supersymmetry-preserving boundary conditions. We finally comment on the relevance and interpretation of our results within the AdS-CFT correspondence.

---

\*hollands@theorie.physik.uni-goe.de

†marolf@physics.ucsb.edu

# 1 Introduction

In a classical system defined by a Hamiltonian or Lagrangian, there typically exists a constant of the motion for any symmetry of the system. Furthermore, any such conserved quantity generates the action of the symmetry on the phase space of the theory. While this result is simple and straightforward to derive for mechanical systems with a finite number of degrees of freedom, the situation can be considerably more complicated in field theories. This is in particular the case for field theories on manifolds with asymptotic regions or boundaries, where the existence and form of the conserved quantities will in general depend in a subtle way on the precise choice of the phase space of the theory, i.e., the asymptotic or boundary conditions that one chooses to impose upon the fields.

Of particular interest in theoretical physics are theories with local gauge invariance, such as diffeomorphism invariance, Yang-Mills type gauge invariance, or local supersymmetry. For theories with diffeomorphism invariance, a general analysis of the existence of conserved quantities within a “covariant phase space framework” was given by Wald et al. [1, 2, 3, 4], (see also [5, 6, 7]). In these works, a general criterion was derived stating when a given boundary or asymptotic condition on the fields allows the existence of conserved charges generating asymptotic spacetime symmetries. These ideas were applied e.g. in [8] to derive formulae for the Bondi energy in higher dimensional general relativity with asymptotically flat asymptotic conditions, and in [9, 10, 11, 12] to derive expressions for bosonic conserved charges (such as energy) in gravity theories with asymptotically AdS boundary conditions.

In this paper, we generalize the analysis of Wald et al. to charges of fermionic type. Our first result is a general criterion for when a given boundary or asymptotic condition on the fields permits the definition of conserved asymptotic charges conjugate to a fermionic asymptotic symmetry. That condition is analogous to the one for bosonic symmetries and can be simply stated as follows: (1) The symplectic flux through the boundary (or at infinity) has to vanish under the imposed boundary (or asymptotic) conditions. (2) The boundary (or asymptotic) conditions must be invariant under the asymptotic symmetry.

We then apply this general formalism to supergravity theories with a negative cosmological constant, for which asymptotically AdS boundary conditions are possible. Our first example is minimal  $\mathcal{N} = 1$  supergravity [13]. Within the usual framework where the boundary metric is held fixed, we show that there exists a unique boundary condition on the metric and the spin-3/2 field so that conditions (1) and (2) are satisfied for local supersymmetry transformations. We also derive the expression for the corresponding supercharge, which is found to agree with an expression derived earlier by a somewhat different method [14].

Next, we investigate extended  $\mathcal{N} = 4$  supergravity [15] in a scalar reduction [16] in which, besides the gravity multiplet, only the scalar multiplet is kept. The scalar fields

have a mass  $m^2 = -2$ , which is within the Breitenlohner-Freedman range [17, 18]. The reduced theory has a local  $\mathcal{N} = 1$  invariance, and we investigate possible boundary conditions such that there exist corresponding Hamiltonian generators for the  $\mathcal{N} = 1$  supersymmetry transformations. Here, we find that there is a 1-parameter family of boundary conditions on the fields in the gravity and scalar multiplets such that (1) and (2) are satisfied. Therefore, conserved generators exist for those boundary conditions. The boundary conditions found by [17] correspond to 2 particular extreme values of the parameter. We also investigate boundary conditions preserving only a 2-dimensional subspace of the fermionic  $\mathcal{N} = 1$  symmetry generating asymptotic Poincare transformations in the ‘‘Poincare compactification’’ of AdS-spacetime. Here, we find that there exist additional boundary conditions satisfying (1) and (2). These boundary conditions can be formulated as a ‘‘supersymmetric Lagrange submanifold condition,’’ in terms of an arbitrarily chosen ‘‘boundary superpotential’’ of suitably defined ‘‘boundary superfields,’’ which in turn are built from the asymptotic values of the fields in the scalar multiplet (in a somewhat non-obvious way). These boundary conditions preserve an  $O(2, 1) \times \mathbb{R}^3$  of the standard  $O(3, 2)$  bosonic symmetry group.

Finally, we construct the explicit expressions for the bosonic and fermionic conserved charges in the scalar reduction of the  $\mathcal{N} = 4$  extended supergravity theory. We find that the expression for the bosonic charges contains a piece that is a surface integral at infinity of the electric Weyl tensor, as well as an extra piece involving the fields in the scalar multiplet. In the case of  $O(2, 1) \times \mathbb{R}^3$  invariant boundary conditions, the extra piece involves the boundary superpotential of the fields in the scalar multiplet. In the special case of the boundary conditions found by Breitenlohner and Freedman [17], only the Weyl piece remains.

Our results have the following standard interpretation in the context of the AdS/CFT-correspondence [19, 20, 21, 22]. In the  $\mathcal{N} = 4$ -theory, changing the boundary conditions corresponds to adding a term to the action of the boundary quantum field theory. Since our boundary conditions are formulated in terms of a supersymmetric Lagrange submanifold condition, the additional piece is given precisely [23, 24, 25] by the boundary superpotential of the boundary fields corresponding to the scalar multiplet. Such deformations of AdS/CFT have been of significant interest, see e.g., [26, 27, 28].

## 2 General definition of fermionic charges

Assume we are given a Lagrange density  $\mathbf{L}$ , (viewed as a  $d$ -form) on a  $d$ -dimensional manifold  $M$ . We assume that  $\mathbf{L}$  depends on a metric  $g_{\mu\nu}$ , tensor fields  $\phi_{\mu_1 \dots \mu_k}$ , gauge fields  $A_\mu$ , and mixed spinor-tensor fields  $\psi^{A_1 \dots A_q}_{\nu_1 \dots \nu_p}$ . We shall abbreviate the collection of all dynamical fields by

$$\Phi = (g_{\mu\nu}, \phi_{\mu_1 \mu_2 \dots \mu_k}, A_\mu, \psi^{A_1 A_2 \dots A_q}_{\nu_1 \nu_2 \dots \nu_p}). \quad (2.1)$$

To define spinors we assume that a spin structure exists and that one has been chosen as part of the definition of the theory. In quantum field theory, by the spin-statistics theorem, fields of half odd integer spin must be anticommuting, while fields of integer spin must be commuting. But in classical field theories, there is a choice to be made whether one wants to view the spinor fields as commuting or anti-commuting, and there does not appear to be a good physical justification for either choice. However, studies of the initial value problem in supergravity theory [29, 30] show that such a theory possess a well-posed initial value formulation if spinors are taken to anti-commute, while difficulties seem to arise if one assumes them to be commuting. We shall therefore assume this in our paper, too, i.e., half odd integer spin fields are taken as anti-commuting and integer spin fields are taken to commute. Concretely, this may e.g. be implemented [31] by taking the fields to be valued in an exterior coefficient algebra  $A = \text{Ext}(V)$ , where  $V$  is an auxiliary infinite dimensional vector space, tensored with the appropriate vector bundles corresponding to the vector or spinor nature of the field. From this perspective, a field is then a formal power series

$$\Phi(x) = \sum_{n=0}^{\infty} v_{(n)} \Phi_{(n)}(x), \quad (2.2)$$

where  $v_{(n)}$  is an  $n$ -th exterior power in  $A$ . Furthermore, it is understood that a half odd integer field component of  $\Phi$  only contains terms with odd  $n$  in the series, while a field with integer spin only contains terms with even  $n$ . As shown in [29, 30] for  $\mathcal{N} = 1$  supergravity, solutions to the field equations then exist as formal power series in this algebra. The first term  $\Phi_{(0)}$  is the zeroth order metric, which is seen to satisfy the Einstein equation with all spinors set to 0. Because  $A = A_+ \oplus A_-$  may be decomposed into even (bosonic) and odd (fermionic) elements, we have a corresponding decomposition of field quantities.

We assume that  $\mathbf{L}$  is commuting and does not depend on any non-dynamical background fields, or equivalently, that  $\mathbf{L}$  is fully diffeomorphism covariant, in the sense that<sup>1</sup>

$$f^* \mathbf{L}(\Phi) = \mathbf{L}(f^* \Phi) \quad (2.3)$$

for any diffeo  $f$  on  $M$ . On the space of (smooth) fields, we will consider various operations. Given a functional  $F$  on the space of fields and a smooth 1-parameter family  $\Phi_t$  in the space of fields with  $\Phi_0 = \Phi$ , we denote by

$$\delta F(\Phi) = \frac{d}{dt} F(\Phi_t) \Big|_{t=0} = \left( \int_M \delta \Phi(x) \frac{\delta}{\delta \Phi(x)} \right) F(\Phi) \quad (2.4)$$

---

<sup>1</sup>Note that it does not make to speak about the pull back of a spinor field separately. But it makes sense if we also pull back simultaneously the metric and corresponding spinor bundles with the diffeomorphism. This is what is meant in this formula, and similar other formulae below.

the variational derivative, along the “vector field”  $\delta\Phi = \frac{d}{dt}\Phi_t|_{t=0}$  on field space<sup>2</sup>. We also define the commutator

$$[\delta_1, \delta_2]\Phi = \frac{d^2}{dt ds}[(\Phi_t)_s - (\Phi_s)_t]|_{t=s=0} . \quad (2.5)$$

For example, if  $f_t$  is a 1-parameter family of diffeomorphisms on  $M$  generated by a vector field  $X$ , and  $\Phi_t = f_t^*\Phi$ , then

$$\delta\Phi = \mathcal{L}_X\Phi \quad (2.6)$$

and the commutator  $[\delta_1, \delta_2]$  of variations associated with two vector fields  $X_1, X_2$  is given by the vector field (on field space)  $\mathcal{L}_{[X_1, X_2]}\Phi$ , i.e., the variation associated with the commutator of the spacetime vector fields.

The variation of  $\mathbf{L}$  can always be written in the form

$$\delta\mathbf{L}(\Phi) = \mathbf{E}(\Phi)\delta\Phi + d\boldsymbol{\theta}(\Phi, \delta\Phi) , \quad (2.7)$$

where  $\mathbf{E}$  are the equations of motion (“Euler-Lagrange equations”), and where  $\boldsymbol{\theta}$  is the  $(d-1)$ -form corresponding to the boundary term that would be obtained if the variation of  $\mathbf{L}$  were done under an integral. The (dual) symplectic current of the theory  $\boldsymbol{\omega}$  is defined as the second anti-symmetrized variation

$$\boldsymbol{\omega}(\Phi, \delta_1\Phi, \delta_2\Phi) = \delta_1\boldsymbol{\theta}(\Phi, \delta_2\Phi) - \delta_2\boldsymbol{\theta}(\Phi, \delta_1\Phi) - \boldsymbol{\theta}(\Phi, [\delta_1, \delta_2]\Phi) . \quad (2.8)$$

It is a simple consequence of the above definitions that, if the fields  $\Phi$  are solutions to the equations of motion, and the variations  $\delta_1\Phi$  and  $\delta_2\Phi$  are solutions to the linearized equations of motion, then  $\boldsymbol{\omega}$  is closed,  $d\boldsymbol{\omega} = 0$ . Given a  $d-1$ -dimensional submanifold  $\Sigma$  of  $M$ , one defines the associated symplectic form by

$$\sigma_\Sigma(\Phi; \delta_1\Phi, \delta_2\Phi) \equiv \lim_{K \uparrow M} \int_K \boldsymbol{\omega}(\Phi; \delta_1\Phi, \delta_2\Phi) , \quad (2.9)$$

where  $K$  is an increasing sequence of compact subsets of  $\Sigma$  whose union is  $\Sigma$ . If  $\Sigma$  is not compact, suitable boundary conditions have to be imposed on the fields in order to ensure that the symplectic form is finite. One might also have to impose restrictions on the possible way of choosing the sequence of compact sets  $K$  to ensure that the limit is unique. If  $\sigma_\Sigma$  converges, then it is independent under compact cobordant variations of the surface  $\Sigma$  in the sense that, if  $K$  is a compact subset of  $M$  such that  $\partial K = \Sigma_1 \sqcup (-\Sigma_2)$ , then  $\sigma_1 = \sigma_2$  for “on-shell variations”, i.e.,  $\delta_1\Phi$  and  $\delta_2\Phi$  satisfying the linearized equations of motion and boundary conditions. Note, however, that this

---

<sup>2</sup>In the integral, we mean the “left” variational derivative when anticommuting quantities are involved.

need not be the case for non-compact cobordisms. In that case, the independence of  $\sigma_\Sigma$  from the slice delicately depends on the asymptotic conditions imposed on the fields. For example, it fails in general relativity when  $\Sigma_1, \Sigma_2$  are asymptotically null surfaces asymptoting to different “cross sections” at infinity, which is the situation considered in the context of asymptotically flat spaces at null infinity.

A diffeomorphism preserving the boundary conditions is called an “asymptotic symmetry”. Since diffeomorphisms leave the Lagrangian invariant, one may ask whether one can define “conserved charges” in the phase space of the theory corresponding to the asymptotic symmetries. They should be defined as the Hamiltonian generators conjugate to a symmetry variation vector field  $\delta\Phi = \mathcal{L}_X\Phi$ , i.e., they should satisfy

$$\delta\mathcal{H}_X = \sigma_\Sigma(\delta\Phi, \mathcal{L}_X\Phi) = \int_\Sigma \omega(\Phi, \delta\Phi, \mathcal{L}_X\Phi) \quad \forall \delta\Phi \quad \text{on shell.} \quad (2.10)$$

The existence of the  $\mathcal{H}_X$  is not automatically guaranteed by the diffeomorphism invariance of the Lagrangian, but depends on the precise choice of the boundary conditions and of  $\Sigma$  in a subtle way. To analyze the question of existence, consider (following [2, 1, 4]) the consistency relation resulting from the fact that the second anti-symmetrized second variation of any quantity should vanish, so we should have

$$\delta_1(\delta_2\mathcal{H}_X) - \delta_2(\delta_1\mathcal{H}_X) - [\delta_1, \delta_2]\mathcal{H}_X = 0 \quad \text{for all } \delta_1, \delta_2. \quad (2.11)$$

Here, the last term takes into account the possibility of having non-commuting variations; it vanishes when  $\delta_1, \delta_2$  are commuting variations (e.g., corresponding to shifts of the phase space coordinates). As a consequence of eq. (2.8), we have the relation

$$\begin{aligned} 0 &= \delta_1\omega(\Phi, \delta_2\Phi, \delta_3\Phi) + \delta_2\omega(\Phi, \delta_3\Phi, \delta_1\Phi) + \delta_3\omega(\Phi, \delta_1\Phi, \delta_2\Phi) \\ &+ \omega(\Phi, \delta_1\Phi, [\delta_2, \delta_3]\Phi) + \omega(\Phi, \delta_2\Phi, [\delta_3, \delta_1]\Phi) + \omega(\Phi, \delta_3\Phi, [\delta_1, \delta_2]\Phi). \end{aligned} \quad (2.12)$$

Using this with  $\delta_3\Phi = \mathcal{L}_X\Phi$ , one finds that the consistency condition (2.11) can be expressed as

$$0 = \int_\Sigma \mathcal{L}_X\omega(\Phi, \delta_1\Phi, \delta_2\Phi) \quad (2.13)$$

$$= \int_{\partial\Sigma} X \cdot \omega(\Phi, \delta_1\Phi, \delta_2\Phi). \quad (2.14)$$

In the second line, we have used Stokes theorem, together with the the standard differential forms identity  $\mathcal{L}_X\omega = d(X \cdot \omega) + X \cdot d\omega$ , and the fact that  $\omega$  is closed on shell. The notation “ $\partial\Sigma$ ” indicates any interior actual boundaries, as well as possibly a boundary at “infinity”. In the examples below, we will attach such a boundary  $\partial M = \mathcal{I}$  to our spacetime manifold, and  $\Sigma$  will be a  $(d-1)$ -dimensional

submanifold which meets  $\mathcal{I}$  in a cross section,  $C$ . A sufficient condition for the consistency condition to hold is that

$$X \cdot \omega \upharpoonright_{\partial\Sigma} = 0. \quad (2.15)$$

The vector fields  $X$  for which the above manipulations are expected to give conserved charges are those vector fields whose asymptotic values at conformal infinity leave the boundary conditions imposed upon the fields  $\Phi$  invariant. Typically, the restrictions of those vector fields to the boundary span the tangent space of  $\mathcal{I}$ . Thus, a sufficient condition for the consistency condition (2.11) to hold is that

$$\omega \upharpoonright_{\mathcal{I}} = 0. \quad (2.16)$$

Whether this condition holds depends on both (a) the Lagrangian and (b) the precise form of the boundary conditions. We will analyze this issue in some example theories below. When eq. (2.11) does hold, the conserved charges can be obtained from eq. (2.10) as follows. Choose any path  $\Phi_t$  of solutions from a canonically fixed reference solution  $\Phi_0$  to  $\Phi = \Phi_1$ . Define

$$\mathcal{H}_X(\Phi) = \mathcal{H}_X(\Phi_0) + \int_0^1 \delta\mathcal{H}_X(\Phi_t) dt. \quad (2.17)$$

where  $\delta\Phi$  is the vector field along the path, and where  $\mathcal{H}_X(\Phi_0)$  is defined arbitrarily. Then it follows from the consistency condition (2.11) that the definition of  $\mathcal{H}_X$  is unchanged under smooth deformation of the path, i.e., it only depends (possibly) on the homotopy class of the path in the space of solutions of the theory. Whence, if this space is for example simply connected, then the definition of  $\mathcal{H}_X$  is unique. On the other hand, if it is not simply connected, the  $\mathcal{H}_X$  might be multi-valued and might consequently be defined only on the covering space of the covariant phase space.

Furthermore, if eq. (2.16) holds (implying (2.11)), then  $\mathcal{H}_X$  will automatically be independent of the choice of the surface  $\Sigma$ , i.e., the cut  $C = \partial\Sigma$  where  $\Sigma$  hits the boundary. This follows straightforwardly from Stokes theorem, and the fact that  $d\omega = 0$  on shell.

As we now describe in some detail, these considerations admit a more or less straightforward generalization to symmetries parametrized by arbitrary tensor fields. Let us assume that there exists a symmetry  $s_\eta$  of the Lagrangian  $\mathbf{L}$  for each choice of some spinor or tensor field  $\eta$ . In other words, assume that for each  $\eta$  there exists a vector field  $s_\eta\Phi$  on field space such that

$$s_\eta\mathbf{L}(\Phi) = d\mathbf{B}_\eta(\Phi), \quad (2.18)$$

where  $\mathbf{B}_\eta$  is a  $(d-1)$ -form that is locally constructed out of the fields and  $\eta$ . This includes in particular the case of infinitesimal diffeomorphism invariance, where  $\eta$

is simply given by a vector field  $\eta = X^\mu \partial_\mu$ , and where  $s_X \Phi = \mathcal{L}_X \Phi$ , and  $\mathbf{B}_X = X \cdot \mathbf{L}$ . More generally, it covers e.g. local gauge transformations, and common fermionic symmetries such as supersymmetry. The precise form of the symmetry is unimportant for our present discussion. For definiteness, let us assume that  $\eta$  is an (anti-commuting) spinor field; this is the case for supersymmetry, which is the context of primary interest in this paper.

We would like to define corresponding Hamiltonian generators in the phase space of the theory by

$$\delta \mathcal{H}_\eta = \int_\Sigma \boldsymbol{\omega}(\Phi, \delta \Phi, s_\eta \Phi). \quad (2.19)$$

(Recall that, by the general definition, the symmetry variation of a functional  $F$  on field space is given by  $s_\eta F(\Phi) = \frac{d}{dt} F(\Phi + t s_\eta \Phi)|_{t=0}$ ). To see whether the generator exists, we must analyze the consistency of this equation in the same fashion as we did above for diffeomorphisms, by taking a second antisymmetrized variation, see eq. (2.11). We now get the consistency condition

$$0 = \int_\Sigma s_\eta \left[ \boldsymbol{\omega}(\Phi, \delta_1 \Phi, \delta_2 \Phi) \right], \quad (2.20)$$

for all solutions  $\Phi$ , and all on-shell variations. Here  $s_\eta$  is defined to act on variations  $\delta \Phi$  as the linearized symmetry transformation, i.e. the transformation on the tangent space of field space induced by  $s_\eta$ . Since  $d\boldsymbol{\omega} = 0$  it follows that

$$0 = d \left\{ s_\eta \left[ \boldsymbol{\omega}(\Phi, \delta_1 \Phi, \delta_2 \Phi) \right] \right\}, \quad (2.21)$$

for all  $\eta$ . By the fundamental lemma of the calculus of variations<sup>3</sup> [32] it consequently follows that there exists a  $(d-2)$ -form  $\mathbf{Y}_\eta$ , such that

$$s_\eta \left[ \boldsymbol{\omega}(\Phi, \delta_1 \Phi, \delta_2 \Phi) \right] = d\mathbf{Y}_\eta(\Phi, \delta_1 \Phi, \delta_2 \Phi). \quad (2.22)$$

Therefore, by Stokes theorem, a sufficient condition for the consistency of eq. (2.20) is that

$$0 = \int_{\partial \Sigma} \mathbf{Y}_\eta. \quad (2.23)$$

This condition is automatically satisfied if

$$\mathbf{Y}_\eta \upharpoonright_{\mathcal{I}} = 0. \quad (2.24)$$

---

<sup>3</sup>Fundamental Lemma (see e.g., [32]): Let  $\alpha[\psi]$  be a  $p$ -form on an  $n$ -dimensional manifold  $M$ , with  $p < n$ , depending on a set of fields  $\psi$  on  $M$ , such that  $d\alpha[\psi] = 0$  for all  $\psi$ , and such that  $\alpha = 0$  when the non-dynamical fields in  $\psi$  vanish. Then  $\alpha[\psi] = d\beta[\psi]$  for some globally defined form  $\beta$ . Note that it is crucial that the relation  $d\alpha[\psi] = 0$  holds for all  $\psi$ . Otherwise, one can only infer the existence of  $\beta$  locally, but possibly not globally if the topology of  $M$  is non-trivial.

Thus, if this equation holds (and in particular, if the boundary conditions are such that  $\mathbf{Y}_\eta$  is actually finite at  $\mathcal{I}$ ), then charges  $\mathcal{H}_\eta$  will exist, and will be conserved, i.e., do not depend on the particular surface  $\Sigma$  chosen in the definition.

A sufficient condition for  $\int_{\partial\Sigma} \mathbf{Y}_\eta \upharpoonright_{\mathcal{I}} = 0$  can be derived as follows. First, assume that the boundary conditions are such that the consistency condition  $\omega \upharpoonright_{\mathcal{I}} = 0$  holds, which, as we showed above, is a sufficient condition for the bosonic asymptotic symmetries to exist. Now, suppose that the  $\eta$ -variations leave the symplectic form invariant on  $\mathcal{I}$ . Then by (2.22)  $d\mathbf{Y}_\eta \upharpoonright_{\mathcal{I}} = 0$ . Consequently, the integral  $\int_C \mathbf{Y}_\eta(\Phi; \delta_1\Phi, \delta_2\Phi)$  does not depend upon the cut  $C$  at  $\mathcal{I}$ . Consider now perturbations  $\delta\Phi$  in this expression which are compactly supported on the particular slice  $\Sigma$ , i.e., that vanish on the particular cut  $C = \partial\Sigma$ . Then the integral vanishes for that cut, and hence for any cut. Thus, the consistency condition is fulfilled for any variation for which the “initial data” on  $\Sigma$  has compact support. In contexts with timelike conformal boundary (such as asymptotically AdS spacetimes), this implies that this integral vanishes for arbitrary perturbations (satisfying the linearized form of the boundary conditions). Thus, in this situation it follows that *the charges  $\mathcal{H}_\eta$  will exist and will be conserved if*

1. The pull back of the symplectic form  $\omega \upharpoonright_{\mathcal{I}}$  to  $\mathcal{I}$  vanishes identically under the assumed boundary conditions.
2. The symmetry leaves the boundary conditions invariant in the sense that if is a field configuration  $\Phi$  satisfying the boundary conditions, then  $s_\eta\Phi$  satisfies the linearized boundary conditions.

It is a general feature of symmetries  $s_\eta$  parametrized by an arbitrary tensor or spinor field  $\eta$  (such as infinitesimal diffeomorphisms, gauge symmetries, or supersymmetry transformations) that the generating conserved charges can be expressed in terms of surface integrals at infinity. In the present formalism, one may derive this fact by generalizing the argument given by Wald et al. [2, 3, 1, 4] for diffeomorphisms.

As with diffeomorphisms, the starting point is the Noether current, defined by

$$\mathbf{J}_\eta(\Phi) = \boldsymbol{\theta}(\Phi, s_\eta\Phi) - \mathbf{B}_\eta. \quad (2.25)$$

On shell (i.e., when the equations of motion hold), this current is conserved,

$$d\mathbf{J}_\eta = d\boldsymbol{\theta}(\phi, s_\eta\Phi) - d\mathbf{B}_\eta = s_\eta\mathbf{L} - d\mathbf{B}_\eta - \mathbf{E} \cdot s_\eta\Phi = 0. \quad (2.26)$$

By the “fundamental lemma”, since this is true for *all*  $\eta$ , it follows that there exists a  $(d-2)$ -form,  $\mathbf{Q}_\eta$ , which is locally constructed out of the fields, such that

$$d\mathbf{Q}_\eta = \mathbf{J}_\eta. \quad (2.27)$$

We refer to  $\mathbf{Q}_\eta$  as the “fermionic” Noether charge. The integrand of eq. (2.19) may then be written as

$$\omega(\Phi, \delta\Phi, s_\eta\Phi) = \delta d\mathbf{Q}_\eta(\Phi) + \delta\mathbf{B}_\eta(\Phi) - s_\eta\{\boldsymbol{\theta}(\Phi, \delta\Phi)\}, \quad (2.28)$$

where the symmetry variation in the last expressions acts on  $\delta\Phi$  by the linearized symmetry transformations. Since  $s_\eta\Phi$  is a solution to the linearized equations of motion, we have  $d\omega(\Phi, \delta\Phi, s_\eta\Phi) = 0$ , so by the fundamental Lemma of variations  $\omega(\Phi, \delta\Phi, s_\eta\Phi)$  is exact, and there is a  $(d-2)$ -form  $\mathbf{A}_\eta(\Phi, \delta\Phi)$  such that

$$d\mathbf{A}_\eta(\Phi, \delta\Phi) = s_\eta\{\boldsymbol{\theta}(\Phi, \delta\Phi)\} - \delta\mathbf{B}_\eta(\Phi), \quad (2.29)$$

for any solution  $\Phi$  to the equations of motion, and any solution  $\delta\Phi$  to the linearized equations. Therefore, we have

$$\delta\mathcal{H}_\eta = \int_{\partial\Sigma} (\delta\mathbf{Q}_\eta - \mathbf{A}_\eta), \quad (2.30)$$

thus expressing  $\delta\mathcal{H}_\eta$  as a boundary integral as promised. For “off shell” variations  $\delta\Phi$ ,  $\delta\mathcal{H}_\eta$  is not in general a boundary integral, but one can show (see Appendix) that it can always be written as the above boundary integral plus an integral over  $\Sigma$  of suitably defined constraints associated with the symmetry  $s_\eta$ .

The consistency requirement  $\delta_1(\delta_2\mathcal{H}_\eta) - \delta_2(\delta_1\mathcal{H}_\eta) - [\delta_1, \delta_2]\mathcal{H}_\eta = 0$  is now equivalent to

$$0 = \delta_1\mathbf{A}_\eta(\Phi, \delta_2\Phi) - \delta_2\mathbf{A}_\eta(\Phi, \delta_1\Phi) + \mathbf{A}_\eta(\Phi, [\delta_1, \delta_2]\Phi) \quad \text{on } \partial\Sigma \quad (2.31)$$

for all variations  $\delta_1\Phi, \delta_2\Phi$  satisfying the linearized equations of motion. Whether this condition holds again depends on the theory under consideration, and on the choice of the boundary conditions. Assuming that the consistency condition (2.31) holds, the equation for  $\delta\mathcal{H}_\eta$  can be integrated as above in the case of “bosonic” symmetries  $X$ , and, eq. (2.30),  $\mathcal{H}_\eta$  is expressed by a boundary integral on shell. An off-shell expression for  $\mathcal{H}_\eta$  is given in the appendix [see eq. (A.179)], where also various issues related to the uniqueness and gauge invariance of  $\mathcal{H}_\eta$  are discussed in detail.

Let us now consider the special case of bosonic symmetries associated with infinitesimal diffeomorphisms,  $s_X\Phi = \mathcal{L}_X\Phi$ , with  $X$  a suitable vector field so that we may show explicitly that the above procedure reduces to the “covariant phase space method” of Wald et al. In that case, because

$$s_X\mathbf{L} = \mathcal{L}_X\mathbf{L} = d(X \cdot \mathbf{L}) + X \cdot d\mathbf{L} = d(X \cdot \mathbf{L}), \quad (2.32)$$

we have  $\mathbf{B}_X = X \cdot \mathbf{L}$ . On shell, we therefore have  $\delta\mathbf{B}_X = X \cdot \delta\mathbf{L} = X \cdot d\boldsymbol{\theta}$ , and thus, from the defining relation for  $\mathbf{A}_X$ , we get

$$\mathbf{A}_X = X \cdot \boldsymbol{\theta}. \quad (2.33)$$

On shell, we therefore have

$$\delta\mathcal{H}_X = \int_C \delta\mathbf{Q}_X - X \cdot \boldsymbol{\theta} \quad (2.34)$$

for symmetries associated with infinitesimal diffeomorphisms. This is the formula derived by Wald et al. [2, 3, 4].

### 3 $\mathcal{N} = 1$ minimal SUGRA with cosmological constant

We now illustrate the above algorithm in the example theory of  $\mathcal{N} = 1$  supergravity with a negative cosmological constant [13] (treated also in the paper of Henneaux and Teitelboim [14] using a different technique). The dynamical fields in this theory are a metric and an anti-commuting spin-(3/2) Majorana field,

$$\Phi = (g_{\mu\nu}, \psi_\mu). \quad (3.35)$$

The Lagrange 4-form is given by

$$\mathbf{L} = \left[ \frac{1}{2}R + \epsilon^{\lambda\rho\mu\nu} \bar{\psi}_\lambda \Gamma \gamma_\mu D_\nu \psi_\rho + 3\ell^2 + \frac{i}{2} \ell \bar{\psi}_\lambda \gamma^{[\lambda} \gamma^{\rho]} \psi_\rho \right] \boldsymbol{\epsilon}, \quad (3.36)$$

where  $\boldsymbol{\epsilon} = \epsilon_{\mu\nu\sigma\rho}$  is the volume 4-form determined by the metric, and where

$$\Gamma = (i/4!) \epsilon^{\mu\nu\sigma\rho} \gamma_\mu \gamma_\nu \gamma_\sigma \gamma_\rho \quad (3.37)$$

is the analogue of  $\gamma_5$  in curved spacetime. The derivative operator  $D_\mu$  is defined by

$$D_\mu = \nabla_\mu + \frac{i}{8} [\gamma^\sigma, \gamma^\rho] (\bar{\psi}_\sigma \gamma_\mu \psi_\rho + \bar{\psi}_\mu \gamma_\sigma \psi_\rho - \bar{\psi}_\mu \gamma_\rho \psi_\sigma), \quad (3.38)$$

where  $\nabla_\mu = \partial_\mu + C_\mu(g)$  is the spin connection of the metric. The quantity  $R$  is the scalar curvature formed from the curvature 2-form of  $D_\mu$ ,

$$R_{\mu\nu} = [D_\mu, D_\nu] \quad (3.39)$$

and  $\ell$  is a constant.

When the spinor field is zero, the Lagrangian has the standard form of the Einstein action, with a negative cosmological constant  $\Lambda = -3\ell^2$ . Exact AdS spacetime with topology  $\mathbb{R}^3 \times \mathbb{R}$  and line element

$$ds_0^2 = -(1 + \ell^2/r^2)dt^2 + (1 + \ell^2/r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (3.40)$$

is an exact solution of this theory, and it appears reasonable to demand as a boundary condition that the metric of a general solution should asymptotically approach that

of exact AdS for large distances. A convenient and elegant framework for formulating this requirement is that of conformal infinity. This is motivated by the fact that the AdS metric can be written as

$$ds_0^2 = \frac{\ell^2}{\Omega^2} \left[ d\Omega^2 - (1 + \Omega^2/2)dt^2 + (1 - \Omega^2/2)(d\theta^2 + \sin^2 \theta d\varphi^2) \right] = \Omega^{-2} d\tilde{s}_0^2 \quad (3.41)$$

where  $\Omega$  is related to  $r$  by a simple coordinate transformation satisfying  $\Omega \sim r^{-1}$  for large  $r$ . Thus, the AdS metric is conformally related to the “unphysical metric”  $d\tilde{s}_0^2$ . The point is now that the unphysical metric can be smoothly extended to a manifold with boundary  $\tilde{M} = M \cup \mathcal{I}$ , where the infinity  $\partial M = \mathcal{I} \cong \mathbb{R} \times S^2$  is represented by the points labelled by  $\Omega = 0$ . The metric induced on  $\mathcal{I}$  is that of the Einstein static universe (ESU),  $\ell^2[-dt^2 + d\theta^2 + \sin^2 \theta d\varphi^2]$ . This motivates one to impose the following asymptotically AdS boundary conditions on the metric:

**Boundary conditions on  $g_{\mu\nu}$ :**

1. One can attach a boundary  $\mathcal{I} = S^2 \times \mathbb{R}$  to the spacetime manifold  $M$  in such a way that  $\tilde{M} = M \cup \mathcal{I}$  is a manifold with boundary.  $\mathcal{I}$  is called the “conformal infinity”, or scri.
2. On  $\tilde{M}$ , there is a smooth field  $\Omega$ , with the properties that  $\Omega$  is a defining function for  $\mathcal{I}$  (i.e., vanishes there and has non-vanishing gradient), and such that

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad (3.42)$$

is smooth at scri. If

$$\tilde{n}^\mu = \ell^{-1} \tilde{g}^{\mu\nu} \tilde{\nabla}_\nu \Omega. \quad (3.43)$$

then we impose that  $\tilde{n}^\mu$  is a unit normal to the boundary  $\mathcal{I}$  of the unphysical spacetime  $(\tilde{M}, \tilde{g}_{\mu\nu})$ , and we impose that the induced metric on  $\mathcal{I}$  is isometric to that of the Einstein static universe  $\mathbb{R} \times S^2$ . We further impose that the extrinsic curvature of  $\mathcal{I}$ ,  $\tilde{K}_{\mu\nu} = \tilde{\nabla}_{(\mu} \tilde{n}_{\nu)} = 0$  at  $\mathcal{I}$ , and that  $\Omega^{-1} \tilde{K}_{\mu\nu}$  is smooth. These conditions are equivalent to demanding that

$$ds^2 = \frac{\ell^2}{\Omega^2} \left[ d\Omega^2 - dt^2 + d\theta^2 + \sin^2 \theta d\varphi^2 \right] + O(\Omega^0) \quad (3.44)$$

Here, the notation  $t_{\mu\nu\dots\sigma} = O(\Omega^a)$  for a tensor field means that  $\Omega^{-a} t_{\mu\nu\dots\sigma}$  can be extended to a smooth tensor field  $\tilde{t}_{\mu\nu\dots\sigma}$  on  $\tilde{M}$ .

As we will see below, Einstein’s equations imply more stringent constraints upon the asymptotic form of the metric in addition to the above boundary conditions. From now on, we will set

$$\ell = 1. \quad (3.45)$$

An asymptotic (bosonic) symmetry is a diffeomorphism  $f$  having the property that  $f^*ds^2$  satisfies our asymptotic conditions whenever the metric  $ds^2$  does. Evidently, if  $f$  acts like a conformal isometry on the ESU with conformal factor  $k^2$ , then we may change  $\Omega$  to  $k\Omega$ , and the unphysical metric  $d\tilde{s}^2$  associated with that new conformal factor will satisfy our asymptotic conditions. Thus, the group of asymptotic symmetries contains the conformal isometry group  $O(3, 2)$  of the Einstein static universe. It is customary to consider as the true physical symmetry group the group of all diffeos  $f$  as described above, factored by the the group of “pure gauge” transformations, which is the group of all  $f$  leaving  $\mathcal{I}$  pointwise fixed. With this definition, the group of asymptotic symmetries is isomorphic to  $O(3, 2)$ .

Since we are in  $d = 4$  dimensions, one may also impose boundary conditions [18] which have slower fall-off near  $\mathcal{I}$ . However, AdS/CFT arguments [33, 34, 35] suggest that such boundary conditions are related to “dual” gravitational theories in spacetimes with compact Cauchy surfaces. As a result, we expect that with such slower fall-off boundary conditions all asymptotic symmetries will be gauge. This result is also suggested by the analysis of [36]. In particular, we expect that such boundary conditions will not lead to interesting conserved charges<sup>4</sup>, and we will not consider them.

The Lagrangian  $\mathbf{L}$  is diffeomorphism invariant. It is also invariant under supersymmetry transformations of the form

$$s_\eta g_{\mu\nu} = -2i\bar{\eta}\gamma_{(\mu}\psi_{\nu)} \quad (3.46)$$

$$s_\eta \psi_\mu = \left( D_\mu + \frac{1}{2}\gamma_\mu \right) \eta, \quad (3.47)$$

in the sense that  $s_\eta \mathbf{L} = d\mathbf{B}_\eta$  for some 3-form  $\mathbf{B}_\eta$ , for any Majorana field  $\eta$ . We would like to construct corresponding generators  $\mathcal{H}_\eta$  on the covariant phase space. According to the general recipe presented in the previous section, the existence of such generators will follow if we impose boundary conditions on the fields  $(g_{\mu\nu}, \psi_\mu)$  such that (1) and (2) hold. We have already chosen boundary conditions for the metric  $g_{\mu\nu}$ , but we have as yet not chosen any boundary conditions on the spinor fields  $\psi_\mu$ . We will now choose them in such a way that (1) and (2) are satisfied.

Condition (1) requires us to know the symplectic current of the theory. It is given by

$$\omega_{\mu\nu\sigma} = \epsilon_{\mu\nu\sigma\rho} * \omega^\rho, \quad (3.48)$$

---

<sup>4</sup>For  $d = 4, 5, 6$ , it is possible [18] to introduce a further class of boundary conditions (termed ‘hybrid boundary conditions’ in [35] by introducing a preferred direction field on  $\mathcal{I}$ . Since this preferred direction field will break supersymmetry, we will not explore such boundary conditions here.

where

$$\begin{aligned} * \omega^\mu &= P^{\alpha\beta\gamma\tau\sigma\mu} [\delta_1 g_{\alpha\beta} \delta_2 C_{\gamma\tau\sigma} - (1 \leftrightarrow 2)] \\ &+ \epsilon^{\mu\alpha\beta\gamma} [\delta_1 \bar{\psi}_\alpha \Gamma_{\gamma\beta} \delta_2 \psi_\gamma + \frac{1}{2} \delta_1 g_{\beta\tau} \bar{\psi}_\alpha \Gamma_{\gamma\tau} \delta_2 \psi_\gamma - (1 \leftrightarrow 2)], \end{aligned} \quad (3.49)$$

and where

$$P^{\alpha\beta\gamma\tau\sigma\mu} = -\frac{1}{2} g^{\alpha\beta} g^{\gamma\tau} g^{\mu\sigma} + \frac{1}{2} g^{\alpha\mu} g^{\gamma\tau} g^{\beta\sigma} + \frac{1}{2} g^{\alpha\gamma} g^{\beta\tau} g^{\mu\sigma}. \quad (3.50)$$

In these formulae,  $\delta C_{\mu\nu\sigma}$  is a shorthand for

$$\delta C_{\mu\nu\sigma} = \delta C_{\mu\nu\sigma}(g) - \frac{i}{2} \delta(\bar{\psi}_\nu \gamma_\mu \psi_\sigma + \bar{\psi}_\mu \gamma_\nu \psi_\sigma - \bar{\psi}_\mu \gamma_\sigma \psi_\nu) \quad (3.51)$$

where  $\nabla_\mu = \partial_\mu + C_\mu(g)$  is the spin connection of  $g_{\mu\nu}$ .

It can be seen that the boundary conditions imposed upon  $g_{\mu\nu}$  imply that the bosonic contribution to the symplectic form is finite at  $\mathcal{I}$ , and that the corresponding contribution to the symplectic flux vanishes. If the spinor field  $\psi_\mu$  is of order  $\Omega^{1/2}$ , then the contribution to the symplectic form from the spinor contribution to  $\delta C_{\mu\nu\alpha}$  is of order  $\Omega^2$  and gives no contribution to the symplectic flux. The finiteness of the contribution quadratic in the variation of the spin 3/2-field to the symplectic form at infinity requires the spinor field  $\delta\psi_\mu$  to fall off at least as fast as  $\Omega^{1/2}$ . From a geometric point of view, it is thus natural to require that the field  $\Omega^{-1/2}\psi_\mu$  be finite and smooth at  $\mathcal{I}$ . The vanishing of the spinor contribution to the symplectic flux through  $\mathcal{I}$ , given by

$$F_{spin-3/2} = \int_{\mathcal{I}} \overline{\delta_1 \psi_\mu} \gamma_\nu \Gamma \delta_2 \psi_\sigma dx^\mu \wedge dx^\nu \wedge dx^\sigma, \quad (3.52)$$

is not ensured however by merely requiring that  $\Omega^{-1/2}\psi_\mu$  be smooth at  $\mathcal{I}$ . A sufficient condition is obviously that the pull-back to  $\mathcal{I}$  of the integrand vanishes,

$$\overline{\delta_1 \psi_{[\mu} \gamma_\nu \Gamma \delta_2 \psi_{\sigma]}} \upharpoonright \mathcal{I} = 0. \quad (3.53)$$

We now fix a boundary condition on  $\psi_\mu$  which does precisely this.

**Boundary condition for  $\psi_\mu$ :**

$$\psi_\mu = O(\Omega^{3/2}). \quad (3.54)$$

We next need to see whether our boundary conditions on  $(g_{\mu\nu}, \psi_\mu)$  are compatible with supersymmetry. On exact AdS-spacetime, one can choose a basis of covariantly constant Majorana spinors, i.e., global solutions to the equation

$$\left( \overset{\circ}{\nabla}_\mu + \frac{1}{2} \overset{\circ}{\gamma}_\mu \right) \eta = 0, \quad (3.55)$$

where the circle above derivative operator and gamma-matrix indicate that the quantities associated with exact AdS spacetime are meant in this equation. Defining the “chiral projections”

$$\tilde{P}_\pm = \frac{1}{2}(1 \pm \tilde{n}^\mu \tilde{\gamma}_\mu), \quad (3.56)$$

(so that  $\tilde{P}_\pm^2 = \tilde{P}_\pm$ ,  $\tilde{P}_+ \tilde{P}_- = 0$  everywhere), one may show that, in any spacetime, one has

$$\left( \nabla_\mu + \frac{1}{2} \gamma_\mu \right) (\Omega^{-1/2} \tilde{\psi}) = \Omega^{1/2} \left( \tilde{\nabla}_\mu + \Omega^{-1} \tilde{\gamma}_\mu \tilde{P}_- \right) \tilde{\psi}, \quad (3.57)$$

for any spinor field  $\psi$ . Thus, taking  $\psi = \eta$ , and using the Killing-spinor equation, we must have that  $\tilde{P}_- \tilde{\eta} \upharpoonright \mathcal{I} = 0$ . In fact, one can show that

$$\tilde{\eta} = \Omega^{1/2} \eta = \tilde{P}_+ \tilde{U} + \frac{1}{2} \Omega \tilde{P}_- \tilde{U} + O(\Omega^2) \quad (3.58)$$

for certain spinor fields  $\tilde{U}$  [14, 17] which are smooth at scri. On a generic spacetime that is only asymptotically AdS, there do not exist any spinors satisfying eq. (3.55). However, using the asymptotic form of the metric and the fact that  $\eta$  is a Killing spinor of AdS, one may show

$$\begin{aligned} \Omega^{-3/2} \left( \nabla_\mu + \frac{1}{2\ell} \gamma_\mu \right) \eta &= \Omega^{-1} \tilde{\gamma}^\nu (\Omega^{-1} \tilde{K}_{\mu\nu}) \tilde{P}_+ \tilde{U} \\ &\quad - \tilde{\gamma}^\nu (\Omega^{-1} \tilde{K}_{\mu\nu}) \tilde{P}_- \tilde{U} + \tilde{\gamma}^\nu \tilde{\nabla}_\nu (\Omega^{-1} \tilde{K}_{\mu\alpha}) \tilde{\gamma}^\alpha \tilde{P}_- \tilde{U} \\ &\quad - \frac{1}{2} \tilde{\nabla}^\nu (\Omega^{-1} \tilde{K}_{\mu\nu}) \tilde{P}_+ \tilde{U} + \dots \end{aligned} \quad (3.59)$$

where dots stands for terms of order  $O(\Omega)$ . In these formulae,  $\tilde{K}_{\mu\nu} = \tilde{\nabla}_{(\mu} \tilde{n}_{\nu)}$  is the extrinsic curvature of  $\mathcal{I}$  with respect to the unphysical metric. Note that our boundary conditions on the metric require that  $\Omega^{-1} \tilde{K}_{\mu\nu}$  is smooth at  $\mathcal{I}$ . In deriving the above equation, the formula

$$\delta(\nabla_\mu \psi) = -\frac{i}{8} [\gamma^\alpha, \gamma^\beta] (\nabla_\alpha \delta g_{\mu\beta} - \nabla_\beta \delta g_{\alpha\mu}) \psi \quad (3.60)$$

for the linearized spin connection was found to be useful.

We can now analyze the question of whether our boundary conditions are left invariant by the supersymmetry transformations. Consider first the variation  $s_\eta \psi_\mu$ , with  $\eta$  a Killing spinor of exact AdS spacetime. From our boundary condition on  $\psi_\mu$ , we know that

$$D_\mu - \nabla_\mu = \frac{i}{8} [\gamma^\nu, \gamma^\sigma] (\bar{\psi}_\nu \gamma_\mu \psi_\sigma + \bar{\psi}_\mu \gamma_\nu \psi_\sigma - \bar{\psi}_\mu \gamma_\sigma \psi_\nu) = O(\Omega^4). \quad (3.61)$$

Thus,  $(D_\mu + \frac{1}{2}\gamma_\mu)\eta$  is given by eq. (3.59), up to higher order terms. However, as it stands, it follows from (3.59) only that

$$\left(D_\mu + \frac{1}{2}\gamma_\mu\right)\eta = \Omega^{1/2}\tilde{\gamma}^\nu(\Omega^{-1}\tilde{K}_{\mu\nu})\tilde{P}_-\tilde{U} + O(\Omega^{3/2}) \quad (3.62)$$

and thus only that  $s_\eta\psi_\mu = O(\Omega^{1/2})$ , instead of our desired boundary condition  $s_\eta\psi_\mu = O(\Omega^{3/2})$ . However, we will now use the equations of motion to show that in fact  $\Omega^{-1}\tilde{K}_{\mu\nu} = 0$  on  $\mathcal{I}$ , so the desired boundary condition follows for  $s_\eta\psi_\mu$ . For this, it is convenient to make a decomposition

$$\tilde{g}_{\mu\nu} = \tilde{n}_\mu\tilde{n}_\nu + \tilde{h}_{\mu\nu} \quad (3.63)$$

of the unphysical metric into ‘‘Gaussian normal form’’, where  $\tilde{n}^\mu = (\partial/\partial\Omega)^\mu$  is geodesic, and where  $\tilde{h}_{\mu\nu}(\Omega)$  is the induced metric on the surfaces of constant  $\Omega$ . Let  $T_{\mu\nu}$  be the matter stress tensor, containing all terms in Einstein’s equation involving the spin-3/2 field, let  $\tilde{L}_{\mu\nu} = T_{\mu\nu} - (1/3)g_{\mu\nu}T$  and let  $\mathcal{R}_{\mu\nu}$  be the Ricci tensor of the metric  $\tilde{h}_{\mu\nu}$  on the submanifolds of constant  $\Omega$ . Then Einstein’s equation may be decomposed as<sup>5</sup>

$$\begin{aligned} \frac{\partial}{\partial\Omega}\tilde{K}_\alpha{}^\beta &= -\mathcal{R}_\alpha{}^\beta + \tilde{K}\tilde{K}_\alpha{}^\beta + 2\Omega^{-1}\tilde{K}_\alpha{}^\beta + \Omega^{-1}\tilde{K}\delta_\alpha{}^\beta \\ &\quad + \tilde{h}_\alpha{}^\gamma\tilde{h}^{\beta\delta}\tilde{L}_{\gamma\delta} + \frac{1}{2}\tilde{L}\delta_\alpha{}^\beta \end{aligned} \quad (3.64)$$

$$\frac{\partial}{\partial\Omega}\tilde{h}_{\alpha\beta} = -2\tilde{h}_{\beta\gamma}\tilde{K}_\alpha{}^\gamma. \quad (3.65)$$

Here  $\partial/\partial\Omega$  should be understood geometrically as  $\mathcal{L}_{\tilde{n}}$ . The boundary condition on  $\psi_\mu$  now implies that  $\tilde{L}_{\alpha\beta} = O(\Omega)$ , from which it can be seen to follow that even  $\tilde{K}_{\mu\nu} = O(\Omega^2)$ , and not just  $\tilde{K}_{\mu\nu} = O(\Omega)$ , as required by our boundary condition on the metric. Thus, it follows from (3.62) that  $s_\eta\psi_\mu = O(\Omega^{3/2})$ , showing that the supersymmetry variation of the spin-3/2 field satisfies the boundary conditions (on shell).

The supersymmetry transformation of the metric under the above boundary conditions can be checked immediately to be of order

$$s_\eta g_{\mu\nu} = O(\Omega^0). \quad (3.66)$$

Thus, the supersymmetry variation of the metric also satisfies our boundary conditions.

---

<sup>5</sup>Here we follow the notation of [9]. The recurrence scheme dates back at least to [37, 38], and has been much developed (see e.g. [39, 40] and references therein) for the purposes of AdS/CFT.

Of course, it remains to be seen that the boundary conditions are also consistent with the field equations, in the sense that there is a wide class of solutions. For the metric, we have given an argument for this above, assuming that the stress tensor is vanishing sufficiently rapidly. Thus, a wide class of solutions with  $\psi_\mu = 0$  is allowed. For the spin-3/2 field, the issue may e.g. be studied via a linearized perturbation analysis. We shall not undertake a detailed study of such perturbations here, but it is clear from the above that our boundary conditions are satisfied by any solution which can be obtained by acting with a supersymmetry transformation on a member of the above class of  $\psi_\mu = 0$  solutions. This class of solutions is sufficiently broad to define an interesting phase space.

Since the symplectic flux through  $\mathcal{I}$  vanishes, the symplectic structure is finite, and the supersymmetry variations leave the boundary conditions invariant, we know from the general analysis of the previous section that a conserved supercharge will exist. We now follow the procedure described in the previous section to determine what those charges are. The Noether current 3-form  $\mathbf{J}$  is given by

$$\mathbf{J}_\eta(\Phi) = \boldsymbol{\theta}(\Phi, s_\eta\Phi) - \mathbf{B}_\eta(\Phi) = 2\bar{\eta}\Gamma\mathbf{E}, \quad (3.67)$$

where  $(*\mathbf{E})_\mu$  are the equations of motion of the spin 3/2-field. Consequently, the Noether current vanishes identically on shell. Thus, the Noether charge vanishes on shell, too,  $\mathbf{Q}_\eta = 0$ . Using the defining relation (2.29) for  $d\mathbf{A}_\eta$ , one finds

$$(d\mathbf{A}_\eta)_{\mu\nu\sigma} = \delta[\epsilon_{\mu\nu\sigma\alpha}(\nabla_\beta\mathbf{N}^{\alpha\beta})], \quad \mathbf{N}_{\alpha\beta} = \bar{\eta}\gamma_{[\alpha}\gamma_\beta\gamma_\rho]\psi^\rho, \quad (3.68)$$

for any on-shell variation of the fields, and so we can read off

$$\mathbf{A}_\eta = \delta(*\mathbf{N}). \quad (3.69)$$

In particular, we explicitly see that the consistency condition holds. Because the Noether charge vanishes on shell, according to eq. (2.30) the conserved charge associated with  $\eta$  is now given by

$$\mathcal{H}_\eta = \int_{\partial\Sigma} *\mathbf{N} = 2 \int_{\partial\Sigma} (\bar{\eta}\gamma_{[\nu}\gamma_\sigma\gamma_\rho]\psi^\rho)u^{\nu\sigma} d^2S. \quad (3.70)$$

Here,  $u^{\mu\nu}$  is the binormal to  $\partial\Sigma$  (normalized with respect to the physical metric), and  $d^2S$  the induced volume element. The fall-off properties imposed upon the fields as part of the boundary conditions guarantee that this integral is finite. By eq. (A.179), the off-shell Hamiltonian is obtained by adding to the above expression the constraints. Since  $\mathbf{J}_\eta = -2\bar{\eta}\Gamma\mathbf{E}$ , it follows that those are given by  $\mathbf{C} = 2\Gamma\mathbf{E}$ , so the off-shell spinor charge is

$$\mathcal{H}_\eta = 2 \int_\Sigma \bar{\eta} \left[ \gamma_{[\mu}\nabla_\nu\psi_{\sigma]} - \frac{1}{4}T_{\tau[\mu\nu}\gamma^\tau\psi_{\sigma]} \right] \epsilon^{\mu\nu\sigma\alpha}u_\alpha d^3S + 2 \int_{\partial\Sigma} (\bar{\eta}\gamma_{[\nu}\gamma_\sigma\gamma_\rho]\psi^\rho)u^{\nu\sigma} d^2S, \quad (3.71)$$

where  $T_\mu$  is the torsion 1-form of  $D_\mu$ , where  $u^\mu$  is the unit normal to  $\Sigma$ , where  $u^{\mu\nu}$  the normalized bi-normal to  $\partial\Sigma$ , and where  $d^3S$  and  $d^2S$  are the respective integration elements induced from the physical metric. This agrees with the expression found by [14] using a different method<sup>6</sup>.

## 4 Extended supergravity

We would now like to extend the above analysis to extended supergravities [16, 15, 41, 42, 43]. In the extended  $\mathcal{N} = 4$  supergravity theory, the fields are a metric, four spin 3/2 fields  $\psi_\mu^{(a)}$ , 4 Majorana spin 1/2 fields  $\chi^{(a)}$ , a real scalar field  $A$  and a real pseudoscalar field  $B$ , and an  $SO(4)$  gauge connection  $A_\mu^{(a)(b)}$ , where  $a, b = 1, \dots, 4$ . The Lagrangian consists of (3.36) plus a complicated matter Lagrangian, which we shall not write down. The full supersymmetry transformations are likewise very complicated. Fortunately, for the present analysis, only the leading parts will be important, and so we will not need to write out the detailed Lagrangian, and transformation rules.

We would like to pick a set of boundary conditions for the above fields such that the symplectic form is finite, such that the symplectic current through  $\mathcal{I}$  vanishes, and such that the boundary conditions are invariant under  $\mathcal{N} = 1$  supersymmetry. The essence of this problem can be studied in a scalar reduction of the  $\mathcal{N} = 4$  theory, and we will from now on limit ourself to that reduction. In the scalar reduction [16], only the fields  $\chi = d^{(a)}\chi^{(a)}$ ,  $A, B, g_{\mu\nu}$ , and  $\psi_\mu = d^{(a)}\psi_\mu^{(a)}$  are kept, where  $d^{(a)}$  is a given direction in field space in which we want to preserve the  $\mathcal{N} = 1$  supersymmetry, and all other fields are set to zero. The precise Lagrangian, and supersymmetry transformation rules of this reduced theory are as follows, see section 4 of [16]:

$$\mathbf{L} = \mathbf{L}_{min} + \mathbf{L}_{kin} + \mathbf{L}_{int} . \quad (4.72)$$

Here,  $\mathbf{L}_{min}$  is identical to the Lagrangian of minimal supergravity, see eq. (3.36), without the cosmological constant term.  $\mathbf{L}_{kin}$  contains the kinetic terms for the

---

<sup>6</sup>Note, however, that the boundary conditions on  $\psi_\mu$  imposed in [14] are stronger than ours.

matter fields,

$$\begin{aligned}
\mathbf{L}_{kin} = \epsilon \left\{ & -i\bar{\chi}\gamma^\mu\nabla_\mu\chi - \frac{1}{(1-u)^2}[(\nabla_\mu A)\nabla^\mu A + (\nabla_\mu B)\nabla^\mu B] \right. \\
& + \frac{i\sqrt{2}}{1-u}\bar{\psi}_\nu\nabla_\mu(A+i\Gamma B)\gamma^\nu\gamma^\mu\chi \\
& + \frac{i}{1-u}\epsilon^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_\mu\psi_\nu(A\nabla_\rho B - B\nabla_\rho A) \\
& \left. + \frac{5}{2(1-u)}\bar{\chi}\Gamma\gamma^\rho\chi(A\nabla_\rho B - B\nabla_\rho A) \right\}, \tag{4.73}
\end{aligned}$$

where  $u = A^2 + B^2$ . The interaction Lagrangian is

$$\begin{aligned}
\mathbf{L}_{int} = & \frac{3-u}{1-u}\epsilon + \frac{i\sqrt{2}}{(1-u)^{1/2}}\bar{\psi}_\mu\gamma^\mu(A+i\Gamma B)\chi\epsilon \\
& + \frac{i}{2(1-u)^{1/2}}\bar{\psi}^\mu[\gamma_\mu, \gamma_\nu]\psi^\nu\epsilon + \dots \tag{4.74}
\end{aligned}$$

where the dots stand for terms that are dropping off fast at infinity and can be ignored for our purposes. The first term in  $\mathbf{L}_{int}$  provides a negative cosmological constant  $\Lambda = -3$ , and gives a mass  $m_A^2 = -2 = m_B^2$  to linear fluctuations of the scalar fields  $A$  and  $B$  about 0.

The terms in the supersymmetry transformation law for  $(A, B, \chi)$  that will be relevant for us are given by

$$s_\eta\chi = -\frac{1}{\sqrt{2}}[\gamma^\mu\nabla_\mu(A+i\Gamma B) + (A-i\Gamma B)]\eta + \dots \tag{4.75}$$

$$s_\eta A = \frac{i}{\sqrt{2}}\bar{\eta}\chi + \dots \tag{4.76}$$

$$s_\eta B = \frac{-1}{\sqrt{2}}\bar{\eta}\Gamma\chi + \dots \tag{4.77}$$

The dots stand for terms that fall off faster at infinity and are not relevant for the considerations below. For the spin-3/2 field and metric, the supersymmetry variations take the form

$$s_\eta g_{\mu\nu} = -2i\bar{\eta}\gamma_{(\mu}\psi_{\nu)} \tag{4.78}$$

$$s_\eta\psi_\mu = \left( D_\mu + \frac{1}{2(1-u)^{1/2}}\gamma_\mu + \frac{1}{2(1-u)}[A\nabla_\mu B - B\nabla_\mu A]\Gamma \right) \eta + \dots, \tag{4.79}$$

where again, dots stands for terms of higher order in  $\Omega$  that can be neglected for our purposes, and where we have set  $u = A^2 + B^2$ . We will now discuss in detail the boundary conditions for the scalar multiplet.

## 4.1 The scalar multiplet

From the Lagrangian given above, eq. (4.73), one derives the following expression for the symplectic current containing contributions from the fields  $A, B, \chi$ :

$$\begin{aligned}
{}^*\omega_\rho &= -2i\delta_1\bar{\chi}\gamma_\rho\delta_2\chi - 2\delta_1A\nabla_\rho\delta_2A - 2\delta_1B\nabla_\rho\delta_2B \\
&+ i\sqrt{2}\delta_2\bar{\psi}_\nu\delta_1A\gamma_\rho\gamma^\nu\chi\frac{1}{1-u} + i\sqrt{2}\bar{\psi}_\nu\gamma_\rho\gamma^\nu\delta_1A\delta_2\left(\chi\frac{1}{1-u}\right) \\
&+(A \leftrightarrow i\Gamma B) \\
&+ i\delta_2\left(\epsilon^{\lambda\mu\nu}{}_\rho\bar{\psi}_\lambda\gamma_\mu\psi_\nu\frac{1}{1-u}\right)[A\delta_1B - B\delta_1A] \\
&+ \delta_2\left(\bar{\chi}\Gamma\gamma_\rho\chi\frac{5}{2(1-u)}\right)[A\delta_1B - B\delta_1A] \\
&-(1 \leftrightarrow 2). \tag{4.80}
\end{aligned}$$

Linear fluctuations of the scalar fields  $A, B$  have a mass of  $m_A^2 = -2 = m_B^2$ . An analysis of possible boundary conditions for linearized fields on exact AdS spacetime with that mass indicate the following fall-off behavior at the linearized level [17, 18]:

$$\tilde{A} \equiv \Omega^{-1}A = \alpha_A + \Omega\beta_A + \dots \tag{4.81}$$

$$\tilde{B} \equiv \Omega^{-1}B = \alpha_B + \Omega\beta_B + \dots \tag{4.82}$$

We will consider only boundary conditions consistent with (4.81) and (4.82). The contributions to the symplectic current 3-form from the scalar fields are then finite at  $\mathcal{I}$ , as may be seen from the expression

$$\omega_{\mu\nu\sigma} \upharpoonright \mathcal{I} = [\delta_1\alpha_A\delta_2\beta_A + \delta_1\alpha_B\delta_2\beta_B - (1 \leftrightarrow 2)]\tilde{n}^\rho\tilde{\epsilon}_{\rho\mu\nu\sigma} \tag{4.83}$$

for the contribution symplectic current involving the fields  $A$  and  $B$ . The corresponding flux through scri is

$$F_{spin-0} = 2 \int_{\mathcal{I}} [\delta_1\alpha_A\delta_2\beta_A + \delta_1\alpha_B\delta_2\beta_B - (1 \leftrightarrow 2)] d^3S, \tag{4.84}$$

where  $d^3S$  is the integration element on  $\mathcal{I}$  induced by the unphysical metric  $\tilde{g}_{\mu\nu}$ . The flux  $F_{spin-0}$  will not vanish unless there is a relationship between  $(\alpha_A, \alpha_B, \beta_A, \beta_B)$ . One possible relationship that will guarantee the vanishing of the flux is to impose

$$\alpha_A = q\alpha_B, \quad \beta_B = -q\beta_A, \tag{4.85}$$

where  $q \in \mathbb{R} \cup \{\infty\}$ . Each choice of  $q$  corresponds to a particular boundary condition. This defines a Lagrange submanifold in the space  $(\alpha_A, \alpha_B, \beta_A, \beta_B)$  with the symplectic form given above. However, other Lagrange submanifolds may be chosen as well.

Linear perturbations of the spinor field  $\chi$  off of exact AdS spacetime may be used to identify a class of reasonable boundary conditions. Such an analysis indicates that [17]

$$\tilde{\chi} = \Omega^{-3/2}\chi \quad (4.86)$$

should be finite (for example smooth) at  $\mathcal{I}$ . A simple way of seeing this is to note that the linearized field  $\chi$  satisfies a massless Dirac equation on AdS-spacetime. The massless Dirac equation in 4 dimensions is conformally invariant, with conformal weight 3/2. This implies that if  $\chi$  is a smooth solution to the massless Dirac equation in AdS spacetime, then  $\tilde{\chi}$  is a smooth solution on the conformal compactification. The contribution to symplectic current from the spinor field  $\chi$  at  $\mathcal{I}$  is then expressible as

$$\omega_{\mu\nu\sigma} = i[\overline{\delta_1\tilde{\chi}}\tilde{\gamma}^\alpha\delta_2\tilde{\chi} - (1 \leftrightarrow 2)]\tilde{\epsilon}_{\alpha\mu\nu\sigma}. \quad (4.87)$$

This will be finite at  $\mathcal{I}$  under the above choice of boundary condition that  $\tilde{\chi} = \Omega^{-3/2}\chi$  be smooth at  $\mathcal{I}$ . The symplectic flux is given by

$$F_{spin-1/2} = \int_{\mathcal{I}} [\overline{\delta_1\chi_+}\Gamma\delta_2\chi_- - \overline{\delta_1\tilde{\chi}_-}\Gamma\delta_2\tilde{\chi}_+ - (1 \leftrightarrow 2)] d^3S, \quad (4.88)$$

where we have defined positive and negative chiral projections by

$$\chi_+ = \tilde{P}_+\tilde{\chi}, \quad \chi_- = i\tilde{P}_+\Gamma\tilde{\chi} = i\Gamma\tilde{P}_-\tilde{\chi}. \quad (4.89)$$

The spin-1/2 symplectic flux will only vanish if there is a relationship between these projections, for example

$$\chi_- = q\chi_+, \quad (4.90)$$

but, again, more general boundary conditions described by an arbitrary Lagrange submanifold in the space  $(\chi_+, \chi_-)$  would also ensure the vanishing of  $F_{spin-1/2}$ . As above, each  $q$  defines a separate boundary condition. There is in particular no reason at this stage why the  $q$  in this equation must be equal to the  $q$  in eq. (4.85). But we will now see that, if they are equal, then the boundary conditions preserve  $\mathcal{N} = 1$  supersymmetry. For this, we need to find how the supersymmetry transformations act on the boundary values of the fields in the chiral multiplet,  $(\alpha_A, \alpha_B, \beta_A, \beta_B, \chi_+, \chi_-)$ .

To this end, we now choose an asymptotic Killing spinor  $\eta = \Omega^{-1/2}\tilde{\eta}$ , and define the quantities  $\eta_+ = \tilde{P}_+\tilde{\eta}$ , and  $\eta_- = \frac{1}{2}\tilde{P}_-\tilde{\eta}$ , both of which are finite at  $\mathcal{I}$  by the arguments given above around eq. (3.58). We insert this into formulae for the supersymmetry transformations. After some algebra, using the field equations for the fermion field,

we find,

$$s_\eta \alpha_A = \frac{i}{\sqrt{2}} \bar{\eta}_+ \chi_- \quad (4.91)$$

$$s_\eta \alpha_B = \frac{i}{\sqrt{2}} \bar{\eta}_+ \chi_+ \quad (4.92)$$

$$s_\eta \beta_B = \frac{-1}{2\sqrt{2}} \bar{\eta}_- \Gamma \chi_- + \frac{1}{\sqrt{2}} \bar{\eta}_+ \Gamma \tilde{h}^{\mu\nu} \tilde{\gamma}_\mu \tilde{\nabla}_\nu \chi_- \quad (4.93)$$

$$s_\eta \beta_A = \frac{+1}{2\sqrt{2}} \bar{\eta}_- \Gamma \chi_+ - \frac{1}{\sqrt{2}} \bar{\eta}_+ \Gamma \tilde{h}^{\mu\nu} \tilde{\gamma}_\nu \tilde{\nabla}_\mu \chi_- \quad (4.94)$$

$$s_\eta \chi_- = \frac{-1}{\sqrt{2}} \left[ +\alpha_A \eta_+ + i\Gamma \beta_B \eta_+ - \tilde{h}^{\mu\nu} \tilde{\gamma}_\mu \tilde{\nabla}_\nu \alpha_A \eta_+ \right] \quad (4.95)$$

$$s_\eta \chi_+ = \frac{-1}{\sqrt{2}} \left[ +\alpha_B \eta_- - i\Gamma \beta_A \eta_+ - \tilde{h}^{\mu\nu} \tilde{\gamma}_\nu \tilde{\nabla}_\mu \alpha_B \eta_+ \right]. \quad (4.96)$$

Here,  $\tilde{h}_{\mu\nu}$  is the metric of the ESU, meaning that all derivatives in the above formulae are tangent to  $\mathcal{I}$ , and hence well-defined on the boundary fields. It is now easy to see that the above boundary conditions are preserved by supersymmetry. For example  $\alpha_A = q\alpha_B$  and  $\beta_B = -q\beta_A$  immediately imply that  $s_\eta \chi_- = q s_\eta \chi_+$ , which is just the desired boundary condition for the spin-1/2 field. The bosonic part of the asymptotic symmetry group associated with these boundary conditions is isomorphic to  $O(3, 2)$ .

We will now show that there exist further boundary conditions that preserve a subset of the  $\mathcal{N} = 1$  supersymmetry transformations associated with a subgroup  $O(2, 1) \times \mathbb{R}^3$  of  $O(3, 2)$ . For this, we first note that the supersymmetry transformations mix only the boundary fields  $(\alpha_A, \beta_B, \chi_-)$  and  $(\alpha_B, \beta_A, \chi_+)$ , but do not mix fields from one set with fields from the other set. This suggests to combine each set into a boundary superfield, and to try and write the supersymmetry transformations on the boundary fields in terms of a derivative operator on superspace. We will now show that this can indeed be done.

For this, we now consider a 2-dimensional subspace of the 4-dimensional space of the asymptotic Killing spinors  $\eta$  that parametrize our supersymmetry transformations. Recall that in exact AdS-space, there are four real linear independent solutions to the Killing spinor equation (3.55). The corresponding conformally rescaled spinors  $\tilde{\eta} = \Omega^{1/2} \eta$  satisfy the conformally related equivalent equation (3.57). To single out the specific 2-dimensional subspace of Killing spinors on AdS-spacetime, we consider the conformally rescaled Killing-spinor equation (3.57) for the metric  $d\tilde{s}_0^2$ , related to the exact AdS-spacetime  $ds_0^2$  via the conformal factor  $\Omega = z$  defined by writing the AdS metric in Poincare coordinates,

$$ds_0^2 = \frac{1}{z^2} (-dt^2 + dx^2 + dy^2 + dz^2) = \Omega^2 d\tilde{s}_0^2, \quad z \geq 0. \quad (4.97)$$

In this description, AdS-spacetime is mapped to the half space  $\mathbb{R}^3 \times \mathbb{R}_+$  of 4-dimensional Minkowski spacetime. The conformal Killing spinor equation (3.57) in this description (with  $\Omega = z$ ) has as obvious solutions the two linearly independent constant spinors  $\tilde{\varepsilon} = \tilde{\eta}$  in 4-dimensional Minkowski spacetime satisfying  $\tilde{P}_- \tilde{\varepsilon} = 0$ , and  $\tilde{P}_+ \tilde{\varepsilon} = \tilde{\varepsilon}$  where we note that  $\tilde{P}_\pm$  is now simply the projector  $(1 \pm \gamma_z)/2$ , with  $\gamma_z$  the flat Minkowski space gamma-matrix. The two real linear independent Killing spinors  $\varepsilon = \Omega^{-1/2} \tilde{\varepsilon}$  thus obtained on AdS-spacetime generate the 3 translations  $\partial_t, \partial_x, \partial_y$  on the conformal boundary defined by  $z = 0$  under the supersymmetry algebra, i.e.,  $[s_{\varepsilon_1}, s_{\varepsilon_2}] = \mathcal{L}_X$ , where  $X = \bar{\varepsilon}_1 \gamma^\mu \varepsilon_2 \partial_\mu$  is a translation on  $\mathbb{R}^3$ , see also [33, 34, 44].

Consider the boundary fields  $(\alpha_A, \alpha_B, \beta_A, \beta_B, \chi_+, \chi_-)$ , viewed now as fields on  $\mathbb{R}^3$  via the diffeomorphism  $(t, r, \theta, \phi) \rightarrow (t, x, y, z)$ , which maps the conformal infinity  $\mathcal{I} = \mathbb{R} \times S^2$  (minus a generator) to  $\mathbb{R}^3$ . They may be obtained from the boundary fields defined above on  $\mathcal{I}$  via the conformal change

$$\chi_\pm \rightarrow k^{3/2} \chi_\pm \quad (4.98)$$

$$\alpha \rightarrow k \alpha \quad (4.99)$$

$$\beta \rightarrow k^2 \beta, \quad (4.100)$$

followed by the diffeomorphism, where  $k$  is the ratio of the conformal factors in eq. (4.97) and (3.41), and  $\beta$  means either  $\beta_A$  or  $\beta_B$ , etc. For any one of the 2 real linear independent Killing spinors  $\varepsilon$  just described, we have

$$s_\varepsilon \alpha_A = \frac{i}{\sqrt{2}} \bar{\varepsilon} \chi_- \quad (4.101)$$

$$s_\varepsilon \alpha_B = \frac{i}{\sqrt{2}} \bar{\varepsilon} \chi_+ \quad (4.102)$$

$$s_\varepsilon \beta_B = + \frac{1}{\sqrt{2}} \bar{\varepsilon} \Gamma \tilde{h}^{\mu\nu} \tilde{\gamma}_\mu \tilde{\nabla}_\nu \chi_- \quad (4.103)$$

$$s_\varepsilon \beta_A = - \frac{1}{\sqrt{2}} \bar{\varepsilon} \Gamma \tilde{h}^{\mu\nu} \tilde{\gamma}_\nu \tilde{\nabla}_\mu \chi_+ \quad (4.104)$$

$$s_\varepsilon \chi_- = \frac{-1}{\sqrt{2}} \left[ +i \Gamma \beta_B - \tilde{h}^{\mu\nu} \tilde{\gamma}_\mu \tilde{\nabla}_\nu \alpha_A \right] \tilde{\varepsilon} \quad (4.105)$$

$$s_\varepsilon \chi_+ = \frac{-1}{\sqrt{2}} \left[ -i \Gamma \beta_A - \tilde{h}^{\mu\nu} \tilde{\gamma}_\nu \tilde{\nabla}_\mu \alpha_B \right] \tilde{\varepsilon}. \quad (4.106)$$

Note that in this expression  $\tilde{h}_{\mu\nu}$  is now the metric on the conformal boundary  $z = 0$  of  $\mathbb{R}^3$  in (4.97), i.e., the flat, 3-dimensional Minkowski metric. These transformations may be written in terms of standard superfields on 3-dimensional Minkowski spacetime  $\mathbb{R}^3$ . For this, introduce 3 Pauli-matrices  $\sigma_j$ ,  $j = t, x, y$  intrinsic to  $\mathbb{R}^3$ , identify  $\tilde{\varepsilon}, \tilde{\chi}_\pm$  with real 2-component spinors on  $\mathbb{R}^3$  associated with the group  $SO(2, 1)$  of

isometries of the boundary  $\mathbb{R}^3$ , and introduce real Grassmann valued spinor field  $\theta^\alpha$ . Define two real boundary superfields on  $\mathbb{R}^3$  by

$$\Phi_+ = \alpha_A + \bar{\theta}\chi_+ + \frac{1}{2}\theta^2\beta_B, \quad (4.107)$$

$$\Phi_- = \alpha_B + \bar{\theta}\chi_- - \frac{1}{2}\theta^2\beta_A. \quad (4.108)$$

Define a superspace derivative operator as usual by

$$\mathcal{D}_\alpha = i\frac{\partial}{\partial\theta^\alpha} + \sigma^j_{\alpha\delta}\theta^\delta\partial_j. \quad (4.109)$$

Then the SUSY transformations of the boundary fields can be written as

$$s_\varepsilon\Phi_\pm = \frac{1}{\sqrt{2}}\tilde{\varepsilon}^\alpha\mathcal{D}_\alpha\Phi_\pm. \quad (4.110)$$

The above choice of the boundary conditions is simply

$$\Phi_+ = q\Phi_-, \quad (4.111)$$

and this is now manifestly invariant under SUSY transformations with respect to the two linear independent spinors  $\varepsilon$ , since these are given in terms of a superderivative. We note that this formalism is in direct parallel to the three-dimensional superspace formalism used to study an AdS/CFT dual of our discussion in [33, 44].

The most general class of boundary conditions preserving the supersymmetry transformations generated by the two Killing spinors  $\varepsilon$  can now be inferred as follows. The first condition on any boundary condition was that the symplectic flux through  $\mathcal{I}$  vanishes. The combined spin-0 and spin-1/2 contributions can be written in terms of the superfields as

$$F_{spin-0} + F_{spin-1/2} = 2 \int_{\mathcal{I}} [\delta_1\Phi_+\delta_2\Phi_- - \delta_1\Phi_-\delta_2\Phi_+] d^2\theta d^3S. \quad (4.112)$$

It follows from this superspace expression for the symplectic flux that it will vanish if we impose as a boundary condition

$$\Phi_- = \frac{\delta\mathcal{W}(\Phi_+)}{\delta\Phi_+}, \quad (4.113)$$

for some functional  $\mathcal{W}$  of  $\Phi_+$ . Indeed, we then have

$$F_{spin-0} + F_{spin-1/2} = 2 \int \frac{\delta^2\mathcal{W}}{\delta\Phi_+(x_1)\delta\Phi_+(x_2)} [\delta_1\Phi_+(x_1)\delta_2\Phi_+(x_2) - \delta_1\Phi_+(x_2)\delta_2\Phi_+(x_1)] = 0. \quad (4.114)$$

If in addition, the functional dependence of  $\mathcal{W}$  on the boundary fields is of the form

$$\mathcal{W}(\Phi_+) = \int_{\mathcal{I}} \mathcal{F}(\Phi_+) d^2\theta d^3S, \quad (4.115)$$

(i.e.,  $\mathcal{W}$  depends on the boundary fields only through the superfield) then the boundary condition is also invariant under the operation  $\mathcal{D}$ , i.e., under supersymmetry. Thus, we have found that the above ‘‘supersymmetric Lagrange submanifold’’ condition gives a general form for boundary conditions perserving  $\mathcal{N} = 1$  supersymmetry. When expressed in terms of the boundary fields, our supersymmetric boundary conditions are as follows.

**SUSY boundary conditions for scalar multiplet:** Let  $\Omega$  denote the conformal factor in (4.97), so that the conformal boundary is mapped to  $\mathbb{R}^3$ . Define the boundary fields  $\alpha_A, \alpha_B, \beta_A, \beta_B, \chi_+, \chi_-$  by

$$A = \Omega\alpha_A + \Omega^2\beta_A + \dots \quad (4.116)$$

$$B = \Omega\alpha_B + \Omega^2\beta_B + \dots \quad (4.117)$$

$$\chi_+ = \Omega^{-3/2}\tilde{P}_+\chi + \dots \quad (4.118)$$

$$\chi_- = i\Omega^{-3/2}\Gamma\tilde{P}_-\chi + \dots \quad (4.119)$$

Let  $\varepsilon$  be the two Killing spinors that approach the constant spinors on the boundary  $\mathbb{R}^3$ . Then, for any smooth function  $\mathcal{F}(z)$  in one variable, the boundary conditions

$$\alpha_A = \mathcal{F}'(\alpha_B), \quad (4.120)$$

$$\beta_B = -\beta_A\mathcal{F}''(\alpha_B) + \frac{1}{2}\mathcal{F}'''(\alpha_B)\bar{\chi}_+\Gamma\chi_+, \quad (4.121)$$

$$\chi_- = \mathcal{F}''(\alpha_B)\chi_+, \quad (4.122)$$

leave  $\mathcal{N} = 1$  supersymmetries generated by the 2 real linear independent Killing spinors  $\varepsilon$  unbroken. They are the component versions of the supersymmetric Lagrange submanifold condition  $\Phi_- = \delta\mathcal{W}/\delta\Phi_+$ , see eq. (4.115).

It can be seen that the asymptotic symmetry algebra associated with these boundary conditions is the superalgebra associated with the bosonic asymptotic symmetry group  $O(2,1) \times \mathbb{R}^3$  that induces the isometries of the boundary metric  $-dt^2 + dx^2 + dy^2$ , see (4.97). Thus, in particular, conserved bosonic and fermionic generators of this algebra exist on phase space, and can be constructed according to the algorithm described in sec. 2. If we make the special choice  $\mathcal{F}(z) = \frac{1}{2}qz^2$ , so that  $\mathcal{W} = \int \frac{1}{2}q\Phi_+^2 d^2\theta d^3S$ , then we obtain the set of boundary conditions invariant under the larger asymptotic symmetry superalgebra with bosonic part  $O(3,2)$  generated by supersymmetry transformation with arbitrary Killing spinor parameter  $\eta$  that was

already described above. In particular, for  $q = 0$  or  $q = \infty$ , we recover the boundary condition proposed by Breitenlohner and Freedman [17] (see also [45]) as a special case of the more general boundary conditions shown above to preserve  $\mathcal{N} = 1$  supersymmetry.

The above boundary conditions can equivalently be formulated in terms of the boundary fields associated with the conformal completion  $\mathcal{I} = S^2 \times \mathbb{R}$ . The corresponding conditions are obtained from (4.120) by simply making the transformations (4.98). One may also implement the full supersymmetry transformations (4.91) associated with an arbitrary Killing spinor field  $\eta$  by a superspace derivative operator, which would now be of the form  $s_\eta = \eta_+ \partial / \partial \theta + (\eta_+ \sigma^i \theta) \partial_i + 2\eta_- \theta$ . Comparing to the superspace derivative (4.110), there is now an additional term  $2\theta\eta_-$ , and this explains why a general function  $\mathcal{F}$  other than  $\mathcal{F}(z) = \frac{1}{2}qz^2$  no longer gives boundary conditions that are invariant under  $s_\eta$  for arbitrary asymptotic Killing spinors  $\eta$ .

## 4.2 The gravity multiplet

The boundary conditions for the gravity multiplet  $(\psi_\mu, g_{\mu\nu})$  are very similar in nature to those in the minimal supergravity theory discussed above, and indeed are identical for the metric. There does, however, exist a subtle difference for the boundary condition of the spin-3/2 field having to do with the backreaction of the scalar fields, and this arises as follows. Above, we discussed the fact that the vanishing of the symplectic flux will hold if we have eq. (3.53), and this suggested to take  $\psi_\mu = O(\Omega^{3/2})$  as the boundary condition for the spin-3/2 field in the minimal supergravity theory. When we checked whether this boundary condition is also preserved by supersymmetry, we found that the supersymmetry variation  $s_\eta \psi_\mu$  was given by eq. (3.59). A priori, this did not satisfy the desired boundary conditions, which would require a faster fall-off in (3.59) by one power. However, it then followed from the field equation that, in fact  $\Omega^{-1} \tilde{K}_{\mu\nu} = 0$  at  $\mathcal{I}$ , so the predicted faster fall-off condition does indeed hold.

In the extended supergravity theory, the same reasoning does not go through as it stands, because the field equations do not imply any more that  $\Omega^{-1} \tilde{K}_{\mu\nu} = 0$ . The reason for this is that, while the stress tensor vanished at  $\mathcal{I}$  in the minimal supergravity, we now have a non-vanishing contribution arising from the scalar field. Instead, using the same expansion techniques as above around (3.64), together with the boundary conditions for the scalar and spin-1/2 fields, we now infer that

$$\Omega^{-1} \tilde{K}_{\alpha\beta} = -\frac{1}{4}(\alpha_A^2 + \alpha_B^2) \tilde{h}_{\alpha\beta}. \quad (4.123)$$

where  $\tilde{h}_{\mu\nu}$  is the induced metric at  $\mathcal{I}$ , i.e., the metric of the ESU. This does not vanish. Consequently, using this in (4.78), we now get for the susy variation of the spin-3/2

field in the reduced  $\mathcal{N} = 4$  supergravity

$$\begin{aligned}
s_\eta \psi_\mu &= \left( D_\mu + \left[ \frac{1}{2} + \frac{1}{4}u + \dots \right] \gamma_\mu \right) \eta \\
&\quad + \frac{1}{2}(1 + u + \dots)(A\nabla_\mu B - B\nabla_\mu A)\Gamma\eta + \dots \\
&= -\Omega^{1/2} \frac{1}{8}(\alpha_A^2 + \alpha_B^2)\tilde{n}_\mu \tilde{P}_+ \tilde{U} + O(\Omega^{3/2})
\end{aligned} \tag{4.124}$$

where we have again used the asymptotic form of the fields  $A, B$ , and the the metric, see eq. (5.134) below. Thus, unlike in the minimal supergravity theory, the supersymmetry variation of the spin-3/2 field is only of order  $O(\Omega^{1/2})$  even after taking into account the field equations. Thus, if we are to choose supersymmetry invariant boundary conditions, we must impose

**Boundary conditions for  $\psi_\mu$ :** We have  $\psi_\mu = \Omega^{1/2}\tilde{n}_\mu\tilde{\psi}_+ + O(\Omega^{3/2})$ , where  $\tilde{\psi}_+$  is smooth at  $\mathcal{I}$  and satisfies  $\tilde{P}_-\tilde{\psi}_+ = 0$ .

The point is now that, since the slow fall off piece in  $\psi_\mu$  is proportional to  $\tilde{n}_\mu$ , the condition (3.53) for the vanishing of the symplectic flux still holds. The supersymmetry variation of the metric also still satisfies our boundary conditions even with the above weaker boundary condition for the spin-3/2 field. This may be seen e.g. by noting that  $(s_\eta g_{\mu\nu}, s_\eta \psi_\mu)$  satisfy the linearized equations of motion. The linearized version of our above expansion techniques then show that  $s_\eta g_{\mu\nu}$  is finite at  $\mathcal{I}$ , and thus satisfies the linearized boundary conditions for the metric. Alternatively, we may directly calculate using the boundary conditions for the spin-3/2 field that

$$s_\eta g_{\mu\nu} = \Omega^{-1}\tilde{n}_{(\mu}\tilde{X}_{\nu)} + \dots, \tilde{X}_\mu = \overline{\tilde{\psi}_+}\tilde{\gamma}_\mu\tilde{\eta}. \tag{4.125}$$

Noting that  $\tilde{n}^\mu\tilde{X}_\mu = 0$  at scri, and putting  $X^\mu = \frac{1}{2}\Omega^2\tilde{g}^{\mu\nu}\tilde{X}_\nu$ , one can then easily see that

$$\Omega^{-1}\tilde{n}_{(\mu}\tilde{X}_{\nu)} = \mathcal{L}_X g_{\mu\nu} + O(\Omega^0). \tag{4.126}$$

Thus, up to an infinitesimal diffeo the susy variation of the metric is of order  $O(\Omega^0)$ , which is the desired boundary condition for the metric. The additional term  $\mathcal{L}_X g_{\mu\nu}$  can be dealt with in either of two ways: 1) Note that  $\mathcal{L}_X$  changes  $g_{\mu\nu}$  only at order  $O(\Omega^{-1})$ . Thus, one may slightly weaken our original boundary condition allowing a departure from ‘‘Gaussian normal gauge’’ at this order. This makes the recursion describing the expansion in powers of  $\Omega$  somewhat more complicated, but does not significantly change the results. 2) One may take the physical symmetry to be not  $s_\eta$ , but  $s_\eta - \mathcal{L}_X$ . One then readily checks that the  $s_\eta - \mathcal{L}_X$  preserve the boundary conditions on all fields, and that they satisfy the same algebra as the original  $s_\eta$ .

## 5 Expressions for the bosonic and fermionic charges in $\mathcal{N} = 4$ extended supergravity

### 5.1 Bosonic charges

Above, we have formulated boundary conditions for the fields  $A, B, \chi$ , and  $g_{\mu\nu}, \psi_\mu$ , in terms of a free function  $\mathcal{F}$ . The group of (bosonic) symmetries leaving these invariant was  $O(3, 2)$  for the special choice  $\mathcal{F}(z) = \frac{1}{2}qz^2$ , and it was  $O(2, 1) \times \mathbb{R}^3$  for general  $\mathcal{F}$ . These boundary conditions were chosen in such a way that the symplectic flux through  $\mathcal{I}$  vanishes. By the general analysis of Sec. 2, we therefore know that Hamiltonian generators for these symmetries exist. We will now find the explicit expressions for those conserved bosonic charges, starting with the case of generic  $\mathcal{F}(z)$ .

The corresponding Lie algebra of  $O(2, 1) \times \mathbb{R}^3$  of asymptotic bosonic symmetries acts by vector fields  $X$  on  $M$  that are smooth and tangent to  $\mathcal{I}$ , and are Killing fields of the induced metric on  $\mathcal{I}$ . These vector fields correspond to translational symmetries  $X = a_0\partial_t + a_1\partial_x + a_2\partial_y$  or Lorentz transformations if we map the boundary  $\mathcal{I}$  conformally to 3-dimensional Minkowski spacetime, i.e., in the Poincare description (4.97) of the conformal infinity. Following the general prescription explained in Sec. 2, we first need to find the Noether charge corresponding to an infinitesimal symmetry  $X$ . It is given by

$$(\mathbf{Q}_X)_{\mu\nu} = \frac{1}{2}\nabla_\delta X_\rho \epsilon^{\delta\rho}{}_{\mu\nu}. \quad (5.127)$$

The symplectic 1-form potential is given by

$$\boldsymbol{\theta}_{\mu\nu\sigma} = \left[ \frac{1}{2}(\nabla^\rho \delta g_\rho{}^\alpha - \nabla^\alpha \delta g^\rho{}_\rho) - 2\delta A \nabla^\alpha A - 2\delta B \nabla^\alpha B - i\bar{\chi}\gamma^\alpha \delta\chi + \dots \right] \epsilon_{\alpha\mu\nu\sigma}, \quad (5.128)$$

where dots stand for other terms involving the fields  $\psi_\mu, \chi, A, B$  that vanish more rapidly at infinity and are not relevant for our analysis. The conserved charge is then obtained by solving

$$\delta\mathcal{H}_X = \int \delta\mathbf{Q}_X - X \cdot \boldsymbol{\theta}. \quad (5.129)$$

where  $\delta$  is an on-shell perturbation, and the integral is taken over a cut at infinity. To solve this equation for the Hamiltonian generators, we need to know the asymptotic form of the on-shell variations  $\delta g_{\mu\nu}, \delta\psi_\mu, \delta A, \delta B, \delta\chi$ . This is determined by our boundary condition and by the field equations, which impose additional constraints on the form of the asymptotic expansion of the metric. These constraints may be e.g. evaluated using the expansion techniques of [9], see also [11, 12]: The Gauss-Codacci relation

$$\tilde{C}_{\mu\alpha\nu\beta}\tilde{n}^\alpha\tilde{n}^\beta = \mathcal{L}_{\tilde{n}}\tilde{K}_{\mu\nu} - \Omega^{-1}\tilde{K}_{\mu\nu} + \tilde{K}_{\mu\alpha}\tilde{K}^\alpha{}_\nu - \frac{1}{2}\tilde{h}_\mu{}^\alpha\tilde{h}_\nu{}^\beta\tilde{L}_{\alpha\beta} - \frac{1}{2}\tilde{h}_{\mu\nu}\tilde{L}^{\alpha\beta}\tilde{n}_\alpha\tilde{n}_\beta \quad (5.130)$$

relates the Weyl tensor of  $\tilde{g}_{\mu\nu}$  to the extrinsic curvature  $\tilde{K}_{\mu\nu}$  of the  $\Omega = \text{const.}$  surfaces near infinity, and the matter stress tensor, where  $\tilde{L}_{\mu\nu} = T_{\mu\nu} - \frac{1}{3}g_{\mu\nu}T^\alpha{}_\alpha$ . Combined with the recursion relations (3.64), this gives

$$(h_{\mu\nu})_3 = -\frac{2}{3}(E_{\mu\nu})_0 + \frac{2}{3}(K_{\mu\alpha}K^\alpha{}_\nu)_1 - \frac{1}{3}(\tilde{h}_\mu{}^\alpha\tilde{h}_\nu{}^\beta\tilde{L}_{\alpha\beta} + \tilde{h}_{\mu\nu}\tilde{L}^{\alpha\beta}\tilde{n}_\alpha\tilde{n}_\beta)_1. \quad (5.131)$$

Here, the subscripts indicate the order in  $\Omega$ , and

$$E_{\mu\nu} = \Omega^{-1}\tilde{C}_{\mu\alpha\nu\beta}\tilde{n}^\alpha\tilde{n}^\beta \quad (5.132)$$

is the leading order part of the electric Weyl tensor. Using the asymptotic expansions of  $A, B, \chi$  near infinity, and the explicit form of the Lagrangian given above results in

$$(\tilde{h}_\mu{}^\alpha\tilde{h}_\nu{}^\beta\tilde{L}_{\alpha\beta} + \tilde{h}_{\mu\nu}\tilde{L}^{\alpha\beta}\tilde{n}_\alpha\tilde{n}_\beta)_1 = 2\tilde{h}_{\mu\nu} \left[ \frac{4}{3}(\alpha_A\beta_A + \alpha_B\beta_B) + \frac{i}{2}\bar{\chi}_-\Gamma\chi_+ \right]. \quad (5.133)$$

Using this result, and eqs. (3.64) and (4.123), one can conclude that the metric has the asymptotic expansion

$$\begin{aligned} ds^2 &= \frac{d\Omega^2}{\Omega^2} - \frac{1}{\Omega^2} \left( 1 - \left\{ \frac{1}{2}\alpha_A^2 + \frac{1}{2}\alpha_B^2 - \frac{1}{2} \right\} \Omega^2 \right) dt^2 \\ &+ \frac{1}{\Omega^2} \left( 1 - \left\{ \frac{1}{2}\alpha_A^2 + \frac{1}{2}\alpha_B^2 - \frac{1}{2} \right\} \Omega^2 \right) ds_{\mathbb{S}^2}^2 + \\ &+ \Omega \left\{ -\frac{2}{3}E_{\mu\nu} - 2 \left[ \frac{4}{9}(\alpha_A\beta_A + \alpha_B\beta_B) - \frac{1}{6}\bar{\chi}_-\Gamma\chi_+ \right] \tilde{h}_{\mu\nu} \right\} dx^\mu dx^\nu \\ &\dots \end{aligned} \quad (5.134)$$

Thus, the asymptotic form of the on-shell perturbations for the fields may be assumed to be

$$\begin{aligned} \delta g_{\mu\nu} &= -\frac{2}{3}\Omega\delta E_{\mu\nu} + 2\Omega\delta \left[ \frac{4}{9}(\alpha_A\beta_A + \alpha_B\beta_B) - \frac{i}{6}\bar{\chi}_-\Gamma\chi_+ \right] \tilde{h}_{\mu\nu} \\ &- \frac{1}{2}\delta(\alpha_A^2 + \alpha_B^2)\tilde{h}_{\mu\nu} + \dots \end{aligned} \quad (5.135)$$

as well as

$$\delta A = \Omega\delta\alpha_A + \Omega^2\delta\beta_A + \dots \quad (5.136)$$

$$\delta B = \Omega\delta\alpha_B + \Omega^2\delta\beta_B + \dots \quad (5.137)$$

$$\delta\chi = \Omega^{3/2}\delta\chi_+ - i\Omega^{3/2}\Gamma\delta\chi_- + \dots \quad (5.138)$$

for the matter fields. The asymptotic form for the spin 3/2 field may also be written out, but is not needed here. If these expansions are used then one finds that

$$(A\nabla_\mu\delta A + B\nabla_\mu\delta B + \frac{i}{2}\chi\gamma_\mu\delta\chi)\tilde{n}^\mu = \frac{1}{2}\Omega\delta(\alpha_A^2 + \beta_B^2) + \Omega^2\delta(\alpha_A\beta_A + \alpha_B\beta_B) \quad (5.139)$$

$$+ \Omega^2(\beta_A\delta\alpha_A + \beta_B\delta\alpha_A + \frac{1}{2}\bar{\chi}_-\Gamma\delta\chi_+ - \frac{1}{2}\bar{\chi}_+\Gamma\delta\chi_-).$$

The term in the last line is seen to vanish on account of our boundary conditions on the fields  $A, B, \chi$ . Substituting this result and (5.135) into the definitions of  $\mathbf{Q}_X$ , and of  $X \cdot \boldsymbol{\theta}$ , gives the result

$$(\delta\mathbf{Q}_X - X \cdot \boldsymbol{\theta})_{\mu\nu} = \delta E_{\sigma\rho} X^\sigma \tilde{n}^\gamma \tilde{\epsilon}^\rho{}_{\gamma\mu\nu} \quad (5.140)$$

$$+ 2 \left[ \frac{1}{3}\delta(\alpha_A\beta_A + \alpha_B\beta_B) + \frac{1}{2}\delta(\bar{\chi}_+\Gamma\chi_-) \right] X^\sigma \tilde{n}^\rho \tilde{\epsilon}_{\sigma\rho\mu\nu}$$

at infinity. We now again use our boundary conditions on the scalar and spinor fields. It follows that the expression for the conserved charge associated with the symmetry  $X$  is given by

$$\mathcal{H}_X = - \int_C E_{\mu\nu} X^\nu u^\mu d^2S \quad (5.141)$$

$$+ \int_C \left\{ \frac{2}{3}\beta_A[\mathcal{F}'(\alpha_B) - \alpha_B\mathcal{F}''(\alpha_B)] + [\alpha_B\mathcal{F}'''(\alpha_B) - \mathcal{F}''(\alpha_B)]\bar{\chi}_+\Gamma\chi_+ \right\} X^\mu u_\mu d^2S,$$

where  $d^2S$  is now the 2-dimensional integration element on the cross section  $C$  of  $\mathcal{I}$  induced by the unphysical metric, and  $u^\mu$  is the timelike unit normal (normalized with respect to the unphysical metric) to  $C$  within  $\mathcal{I}$ . The first term in  $\mathcal{H}_X$  is the standard gravitational contribution. The second term arises from the backreaction of the matter fields onto the metric and depends upon the function  $\mathcal{F}$  specifying the boundary conditions. This expression simplifies somewhat for the case of  $O(3, 2)$ -invariant boundary conditions  $\mathcal{F} = \frac{1}{2}qz^2$ , in which case the Hamiltonian generators read

$$\mathcal{H}_X = - \int_C E_{\mu\nu} X^\nu u^\mu d^2S - q \int_C \bar{\chi}_+\Gamma\chi_+ X^\mu u_\mu d^2S. \quad (5.142)$$

In this case, the matter term and the gravitational term are separately conformally invariant. It is interesting to note that the scalars make no contribution to (5.142). For the special boundary conditions found by Breitenlohner and Freedman [17] (corresponding to  $q = 0$  or  $q = \infty$ ) the last term disappears as well<sup>7</sup>.

---

<sup>7</sup>To see this in the case  $q = \infty$  one should write the last term in terms of  $\chi_-$ .

## 5.2 Positivity of the energy

We now discuss the positivity properties of our expression for the bosonic charges  $\mathcal{H}_X$  given by eq. (5.141), for timelike asymptotic symmetries  $X$ , i.e., the positivity of the energy of the theory. We first discuss the general boundary conditions on the scalars preserving an  $O(2,1) \times \mathbb{R}^3$  bosonic asymptotic symmetry group, specified by a general smooth function  $\mathcal{F}$  as in (4.120). In that case, as follows from the discussion below (4.97), a future directed timelike asymptotic symmetry corresponds to a vector field  $X$  given by

$$X = a_0 \frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} \quad (5.143)$$

at  $\mathcal{I}$ , where  $(a_0, a_1, a_2) \in \mathbb{R}^3$  is a future directed, timelike vector in 3-dimensional Minkowski spacetime and where  $(t, x, y)$  are the coordinates of  $\mathcal{I}$  in the Poincare description of the conformal boundary, see eq. (4.97). Such a vector field can be written as

$$X = \bar{\varepsilon} \gamma^\mu \varepsilon \quad (5.144)$$

where  $\varepsilon$  is a complex linear combination of the 2 special linearly independent asymptotically Killing spinors<sup>8</sup> described below eq. (4.97). We will now show that

$$\mathcal{H}_X \geq 0 \quad (5.145)$$

for such vector fields, for any solution of the field equations with the stated boundary conditions for which the spinor fields all vanish

$$\psi_\mu = 0 = \chi. \quad (5.146)$$

The condition (5.146) guarantees that  $\mathcal{H}_X$  is real, as opposed to being merely an even element of the exterior (Grassmann) algebra.

The Lagrangian (4.72) then reduces to its bosonic part, given by

$$\mathbf{L}_{bos} = \left[ \frac{1}{2} R - G_{IJ}(\phi) \nabla^\mu \phi^I \nabla_\mu \phi^J - V(\phi) \right] \epsilon. \quad (5.147)$$

Here, the scalar fields  $A, B$  have been combined into a 2-component field  $\phi = (A, B)$  taking values in a 2-dimensional target space with metric and potential given by

$$G_{IJ}(\phi) d\phi^I d\phi^J = \frac{|d\phi|^2}{(1 - |\phi|^2)^2}, \quad V(\phi) = -\frac{3 - |\phi|^2}{1 - |\phi|^2}. \quad (5.148)$$

---

<sup>8</sup>Unlike in the rest of the paper, spinors in this subsection are taken to be commuting. The reason is that we want the spinor charge  $Q_X$  defined below to be a real number. The spinors considered here are not dynamical fields.

The potential arises from the pre-potential  $P(\phi) = (1 - |\phi|^2)^{-1/2}$  in the sense that there holds the equation

$$V = -3P^3 + G^{IJ}\partial_I P\partial_J P, \quad (5.149)$$

which is closely related to the fact that  $\mathbf{L}_{bos}$  comes from a supersymmetric theory. Following [46], generalizing in turn arguments of [47, 48, 49, 51, 50], one can show that the existence of such a prepotential entails the existence of a positive definite quantity  $Q_X \geq 0$ , which will turn out to be equal to  $\mathcal{H}_X$ , thus establishing the desired positivity property. The quantity  $Q_X$  is defined as follows. One first defines an analog of the Nester [50] 2-form, given by

$$F^{\alpha\beta} = \bar{\varepsilon}\gamma^{[\alpha}\gamma^\beta\gamma^{\rho]} \left( \nabla_\rho + \frac{1}{2}P(\phi)\gamma_\rho \right) \varepsilon. \quad (5.150)$$

This quantity will turn out to be closely related to the fermionic supercharges of the theory, but for the moment we do not need this fact. The quantity  $Q_X$  is now defined as

$$Q_X = \int_C {}^*\mathbf{F}. \quad (5.151)$$

Let us choose a spacelike 3-surface intersecting the cut  $C$  of  $\mathcal{I}$  transversally, with no interior boundary (we assume that such a surface can be chosen). Then, applying Stoke's theorem, we have

$$Q_X = \int_\Sigma d({}^*\mathbf{F}) = \int_\Sigma (\nabla^\mu F_{\mu\nu})u^\nu dS, \quad (5.152)$$

where  $dS$  is the natural integration element on  $\Sigma$ , and where  $u^\mu$  is the normal, normalized to -1 here with respect to the physical metric. Using (5.149) and Einstein's equation, one can calculate [46], that

$$(\nabla^\mu F_{\mu\nu})u^\nu = -2(\widehat{\nabla}_i \varepsilon)^\dagger \gamma^i \gamma^j \widehat{\nabla}_j \varepsilon + 2(\widehat{\nabla}_i \varepsilon)^\dagger \widehat{\nabla}^i \varepsilon + G_{IJ}\lambda^{I\dagger}\lambda^J, \quad (5.153)$$

where  $i, j$  are indices associated with the tangent space of  $\Sigma$ , where  $\widehat{\nabla}_\mu = \nabla_\mu + \frac{1}{2}P(\phi)\gamma_\mu$ , and where

$$\lambda^I = \frac{1}{\sqrt{2}}(\gamma^\mu \nabla_\mu \phi^I + 2G^{IJ}\partial_J P)\varepsilon. \quad (5.154)$$

The last 2 terms in eq. (5.153) are manifestly positive, while the first term can be set to 0 by imposing the ‘‘Witten condition’’  $\gamma^i \widehat{\nabla}_i \varepsilon = 0$  on  $\Sigma$  (essentially the spatial Dirac equation with suitable boundary conditions). As one can show using functional analytic techniques [11] (see also [52]) a global, smooth solution to this condition with the desired boundary conditions indeed exists under our assumed boundary conditions. Thus, we have shown that  $Q_X \geq 0$ .

We next need to relate  $Q_X$  to  $\mathcal{H}_X$ . For this, we expand the metric and scalar field as above in (5.134) and (4.116) near  $\mathcal{I}$ , and we additionally expand  $\varepsilon$  near  $\mathcal{I}$  using the fact that it satisfies the Witten condition, see [11, 12] and [52] for similar calculations in an almost identical theory with only one scalar field. Using our boundary conditions on the scalar fields (4.120), as well as an expansion of  $P$  near  $\phi = 0$  one finds<sup>9</sup>

$$(*\mathbf{F})_{\mu\nu} = \tilde{\epsilon}_{\mu\nu\alpha\beta}\tilde{n}^\beta \left[ E^{\alpha\rho} X_\rho - \frac{2}{3}(\alpha_A\beta_A + \alpha_B\beta_B)X^\alpha \right] \quad (5.155)$$

at  $\mathcal{I}$ . Integrating this expression over a cut  $C$  at  $\mathcal{I}$ , again using our boundary conditions on the scalar fields, and comparing the result to our expression (5.141) for  $\mathcal{H}_X$  with the fermionic fields  $\chi$  set to 0, one sees that  $Q_X = \mathcal{H}_X$ . Thus, we have shown that  $\mathcal{H}_X \geq 0$  for timelike asymptotic symmetries and boundary conditions preserving an  $O(2, 1) \times \mathbb{R}^3$  group of asymptotic symmetries.

The choice  $\mathcal{F}(z) = \frac{1}{2}qz^2$  is a special case which, as explained above, corresponds to an asymptotic symmetry group isomorphic to  $O(3, 2)$ . In that case, we have a larger class of timelike asymptotic Killing fields, which can be written as  $X^\mu = \bar{\eta}\gamma^\mu\eta$ , where  $\eta$  is a complex linear combination of the 4 real linearly independent asymptotic Killing spinors rather than only a linear combination of the special 2 real linear independent spinors  $\varepsilon$  described below (4.97). Each such  $\mathcal{H}_X$  is positive by the argument above.

### 5.3 Fermionic charges

Since we have identified a set of boundary conditions preserving  $\mathcal{N} = 1$  supersymmetry and enforcing vanishing symplectic flux through  $\mathcal{I}$ , we know by the analysis of Sec. 2 that corresponding conserved Hamiltonian generators associated with a supersymmetry variation exist. We now explicitly determine their form. As in the previous subsection, it is instructive to treat separately the special boundary conditions defined by  $\mathcal{F}(z) = \frac{1}{2}qz^2$ , preserving a group  $O(3, 2)$  of asymptotic bosonic symmetries, and the general boundary conditions defined by a general  $\mathcal{F}(z)$ , preserving only a subgroup  $O(2, 1) \times \mathbb{R}^3$  of asymptotic bosonic symmetries.

In the general case, asymptotic fermionic symmetries are associated with asymptotic Killing spinors  $\varepsilon$  which are linear combination of the 2 special linearly independent spinors described below eq. (4.97). For such spinors, the asymptotic Killing field  $X^\mu = \bar{\varepsilon}\gamma^\mu\varepsilon$  is a translation on  $\mathcal{I}$ , i.e., of the form (5.143), corresponding to generators in the translation subgroup of  $O(2, 1) \times \mathbb{R}^3$ . To determine the fermionic Hamiltonian generators  $\mathcal{H}_\varepsilon$  following the general procedure outlined in Sec. 2, we must determine the Noether charge,  $\mathbf{Q}_\varepsilon$  [see eq. (2.27)] and the 2-form  $\mathbf{A}_\varepsilon$  [see eq. (2.29)] for extended supergravity, and then determine  $\mathcal{H}_\varepsilon$  via eq. (2.30). From the general form of the

---

<sup>9</sup> Here it is important that  $P$  is a “good” superpotential of the type called “ $P_-$ ” in [52], see that paper for a detailed explanation.

expressions that are involved in these quantities, one can infer that the integrand  $\mathbf{h}_\varepsilon$  of the Hamiltonian generator  $\mathcal{H}_\varepsilon = \int_C \mathbf{h}_\varepsilon$  must be a 2-form satisfying the following general properties: (1) It must be locally constructed out of  $\varepsilon$ , the metric  $g_{\mu\nu}$ , the Levi-Civita tensor  $\epsilon_{\mu\nu\sigma\rho}$ , the gamma-matrices  $\gamma_\mu$ , the spin 3/2-field  $\psi_\mu$ , and the matter fields  $A, B, \chi$ , but not the covariant derivatives of these fields. (2) It must be linear in  $\varepsilon$ . (3) It must be manifestly finite at  $\mathcal{I}$ . These restrictions only leave a relatively small number of possible terms, since for example each factor of  $A$  or  $B$  drops off as  $\Omega$  near infinity, so that terms involving higher powers of  $A$  or  $B$  will automatically vanish at  $\mathcal{I}$ . In fact, it can be seen in this way that possible terms fulfilling (1), (2) and (3) can at most be linear in  $\psi_\mu, \chi, A, B$ , and cannot depend upon  $A, B, \chi$  if they depend upon  $\psi_\mu$ . One possible term fulfilling these criteria is the 2-form  ${}^*\mathbf{N}_\varepsilon$  entering the definition of the Hamiltonian generator in minimal supergravity (see eq. (3.68)). Let us define

$$\mathcal{G}_\varepsilon = \int_C {}^*\mathbf{N}_\varepsilon. \quad (5.156)$$

We will now show that, in fact,

$$\mathcal{H}_\varepsilon = \mathcal{G}_\varepsilon \quad (5.157)$$

i.e., the formula for the Hamiltonian generators for the fermionic charges in extended supergravity is the same as in minimal supergravity.

We could prove this directly by going stubbornly through the definitions of sec. 2, but this is somewhat tedious because the Lagrangian (4.72) and the corresponding supersymmetry transformations are relatively complicated. We will instead use an indirect argument proving that  $\mathcal{H}_\varepsilon = \mathcal{G}_\varepsilon$ . This argument is based on our knowledge of the bosonic charges  $\mathcal{H}_X$  associated with an asymptotic symmetry vector field  $X$ , see eq. (5.141). The point is that, since the generators  $\mathcal{H}_\varepsilon$  and  $\mathcal{H}_X$  are defined as Hamiltonian generators, they form a representation of the superalgebra of asymptotic symmetries (isomorphic to the super Poincaré algebra in 3 dimensions) on the covariant phase space under the Poisson bracket (with respect to the symplectic form  $\sigma$ ), that is,

$$\{\mathcal{H}_{\varepsilon_1}, \mathcal{H}_{\varepsilon_2}\} = \mathcal{H}_X, \quad X^\mu = \bar{\varepsilon}_1 \gamma^\mu \varepsilon_2. \quad (5.158)$$

Thus, the fermionic charges are related to the bosonic ones, whose form is already known from the previous section. That relation will enable us to derive the desired expression for the fermionic generators in extended supergravity (5.157). We will show that

$$\{\mathcal{H}_{\varepsilon_1}, \mathcal{G}_{\varepsilon_2}\} = \int_C s_{\varepsilon_1} {}^*\mathbf{N}_{\varepsilon_2} = \mathcal{H}_X. \quad (5.159)$$

In view of eq. (5.158), this means that  $\mathcal{H}_\varepsilon$  can differ from  $\mathcal{G}_\varepsilon$  at most by a term  $\mathcal{D}_\varepsilon$  with vanishing supersymmetry variation. We now argue that there is no such term. Indeed, by properties (1), (2), and (3), the 2-form integrand of  $\mathcal{D}_\varepsilon$  would have to be a local functional of the dynamical fields (but not their derivatives) that is linear in

$\varepsilon$ , and at most linear in  $\psi_\mu, A, B, \chi$ . The possible terms can be written out explicitly, and it can be seen using eqs. (4.101)-(4.106) that there does not exist a term which has a vanishing supersymmetry variation (up to an exact 2-form). Thus, if we can prove eq. (5.159), then eq. (5.157) follows.

The supersymmetry variation of the 2-form  ${}^*\mathbf{N}_{\varepsilon_2}$  on  $\mathcal{I}$  in expression (5.159) can be computed using the definition of the supersymmetry transformation on the spin-3/2 field, giving at  $\mathcal{I}$

$$\begin{aligned} (s_{\varepsilon_1} {}^*\mathbf{N}_{\varepsilon_2})_{\mu\nu} &= \epsilon^{\alpha\beta}{}_{\mu\nu} \bar{\varepsilon}_1 \gamma_{[\alpha} \gamma_{\beta} \gamma_{\rho]} \left( \nabla^\rho + \frac{1}{2} P(A, B) \gamma^\rho \right) \varepsilon_2 \\ &\quad + (A \nabla_{[\mu} B - B \nabla_{[\mu} A) X_{\nu]}], \end{aligned} \quad (5.160)$$

where  $P$  is the prepotential given above in eq. (5.149). An important thing to note is that the first term on the right side is essentially the dual of the Nester 2-form given above in eq. (5.150). Thus, by essentially the same argument as given there, the integral over a cut  $C$  of the first term will give us  $\mathcal{H}_X$ . We only need to take care that the fields  $\chi, \psi_\mu$  are now not assumed to vanish, and consequently might appear e.g. in the asymptotic form of the metric (5.134). Taking into account this difference, we obtain for the first term in (5.160) the expression

$$(\text{First term}) = \tilde{\varepsilon}_{\mu\nu}{}^{\alpha\beta} \tilde{n}_\beta \left[ E_{\alpha\rho} X^\rho - \frac{2}{3} (\alpha_A \beta_A + \alpha_B \beta_B - \frac{3}{2} \bar{\chi}_+ \Gamma \chi_-) X^\alpha \right]. \quad (5.161)$$

On the other hand, the second term in (5.160) can be expressed as

$$(\text{Second term}) = \tilde{X}_{[\nu} \tilde{\nabla}_{\mu]} \{ 2\mathcal{F}(\alpha_B) - \alpha_B \mathcal{F}'(\alpha_B) \} \quad (5.162)$$

using the boundary conditions (4.120), where  $\tilde{X}_\mu = \tilde{g}_{\mu\nu} X^\nu$  is given by

$$\tilde{X} = a_0 dt + a_1 dx + a_2 dy \quad (5.163)$$

in view of eq. (5.143). We now integrate eq. (5.160) over a cut  $C$  of  $\mathcal{I}$ , and compare the result with our expression for the Hamiltonian generator (5.141) using (5.161) and (5.162). We find

$$\{ \mathcal{H}_{\varepsilon_1}, \mathcal{G}_{\varepsilon_2} \} = \mathcal{H}_X + \int_C d[2\mathcal{F}(\alpha_B) - \alpha_B \mathcal{F}'(\alpha_B)] \wedge \tilde{X}. \quad (5.164)$$

Integrating the last term by parts and computing  $d\tilde{X} = 0$  from (5.163) establishes the desired relation (5.159).

We next consider the case of the special  $O(3, 2)$ -invariant boundary conditions corresponding to the special choice  $\mathcal{F}(z) = \frac{1}{2} q z^2$ . In that case, our supersymmetry transformations  $s_\eta$  are parametrized by arbitrary asymptotic Killing spinors  $\eta$ , and

not the special 2-dimensional subspace of asymptotic Killing spinors  $\epsilon$  giving rise to translational asymptotic Killing fields  $X$ , see (5.143). In that case, eq. (5.164) still holds (with  $\epsilon_i$  replaced by the more general spinors  $\eta_i$ ), but it is no longer true that  $d\tilde{X} = 0$  on  $\mathcal{I}$ , because  $X$  need no longer be a translation on  $\mathcal{I}$ . However, we now have

$$2\mathcal{F}(\alpha_B) - \alpha_B\mathcal{F}'(\alpha_B) = 0. \quad (5.165)$$

Thus, the second term in eq. (5.164) still vanishes, demonstrating that the desired relation (5.159) also holds (with  $\epsilon_i$  replaced by  $\eta_i$  in that equation) for the larger set of supersymmetry generators associated with  $\mathcal{F}(z) = \frac{1}{2}qz^2$ .

## 6 Summary and conclusions

In this paper we have investigated classical field theories with local symmetries, and in particular theories with fermionic local symmetries, i.e., supergravity theories. Let us summarize the main results of this paper:

- In section 2, we have gave a general definition of Hamiltonian generators conjugate to a local gauge symmetry in classical field theories on a manifold, within a covariant phase space framework. The local symmetries were assumed to be either of bosonic type (such as local Yang-Mills type gauge symmetries or diffeomorphisms) or of fermionic type (such as supersymmetry transformations). For theories requiring the specification a set of boundary and/or asymptotic conditions, the existence of the generators is not guaranteed. We derived a simple general criterion stating when such charges exist and are conserved. The criterion is that the symplectic flux through any real or conformal boundary should vanish, and that the boundary conditions should be invariant under the symmetry. In the case of a bosonic symmetry, our method reduces to that of Wald et al. [2, 3, 4]
- We then applied our formalism to two example theories: Minimal ( $\mathcal{N} = 1$ ) supergravity with a negative cosmological constant, and a consistent truncation of extended  $\mathcal{N} = 4$  supergravity with a complex scalar field of mass  $m^2 = -2$ , a Majorana fermion field, and a cosmological constant  $\Lambda = -3$ . For minimal supergravity, we chose boundary conditions such that the metric approaches that of AdS-spacetime for large distances, and in particular such that the boundary metric is held fixed. We then found boundary conditions for the spin-3/2 field such that our criteria for the existence of supercharges were met. In particular, those boundary conditions are invariant under the supergroup of the asymptotic symmetry group  $O(3, 2)$ . An explicit formula for the supercharge

was also derived, and it agreed with the formula previously found by Henneaux and Teitelboim [14] by a somewhat different method.

- For extended  $\mathcal{N} = 4$  supergravity theory, one also has to choose boundary conditions for the matter fields. We found a 1-parameter family of boundary conditions perserving  $\mathcal{N} = 1$  supersymmetry, satisfying our criteria for the existence of the corresponding  $\mathcal{N} = 1$  supercharges. These boundary conditions are, in particular invariant under the  $\mathcal{N} = 1$  supergroup of  $O(3, 2)$ . For particular values of the parameter, our boundary conditions reduce to those found by Breitenlohner and Freedman [17]. We also found a more general class of boundary conditions of non-linear type that are invariant under a subgroup  $O(2, 1) \times \mathbb{R}^3 \subset O(3, 2)$ , and which are also invariant under a restricted set of supersymmetry transformations generating Poincare transformations of the asymptotic boundary  $\mathbb{R}^3$  of spacetime (corresponding to the description of AdS-spacetime as conformal to a half space of 4 dimensional Minkowski space). These boundary conditions are characterized by an arbitrary function,  $\mathcal{F}(z)$ , in one variable. They could be written as a “supersymmetric Lagrange submanifold condition”

$$\Phi_+ = \frac{\delta}{\delta\Phi_-} \int_{\mathcal{I}} \mathcal{F}(\Phi_-) d^3S d^2\theta, \quad (6.166)$$

where  $\Phi_{\pm}$  are boundary superfields on the conformal boundary  $\mathcal{I}$  of spacetime, defined from the asymptotic tails of the scalar and spin-1/2 fields in the theory. The boundary conditions of [17] correspond to the particular choice  $\mathcal{F}(z) = 0$ . Asymptotic supercharges were shown to exist (for general  $\mathcal{F}$ ) that generate the restricted class of supersymmetry transformations. In the context of the AdS/CFT correspondence, the boundary conditions associated with a general  $\mathcal{F}$  correspond to a modified boundary action by general arguments [24, 25, 23],

$$S_{bndy} \rightarrow S_{bndy} + \int_{\mathcal{I}} \mathcal{F}(\mathcal{O}) d^3S d^2\theta, \quad (6.167)$$

with  $\mathcal{O}$  the dual boundary superfield in the CFT. This modification will in general break superconformal invariance, but preserves supersymmetries corresponding to the 3-dimensional Poincare group. Our boundary conditions thus give rise to interesting QFT-duals.

- Finally, we gave formulae for the bosonic and fermionic charges in extended  $\mathcal{N} = 4$  supergravity, and we proved that the energy is always positive when the fermionic fields are set to 0. Our formulae for the fermionic charges agree with those in minimal  $\mathcal{N} = 1$  supergravity, though the explicit expressions for the bosonic charges generally contain an additional term associated with the matter fields. This additional term depends upon the function  $\mathcal{F}$  characterizing the boundary conditions.

Although we have focused on particular supergravity theories, the general pattern of results is clear. In the  $\mathcal{N} = 4$  theory, the freedom to choose a variety of boundary conditions preserving  $\mathcal{N} = 1$  supersymmetry stems from the presence of appropriate matter chiral superfields. One therefore expects a similar set of allowed  $\mathcal{N} = 1$  boundary conditions in any  $d = 4$  theory containing chiral superfields with masses in the usual window [17] above the Breitenlohner-Freedman bound. This result is also suggested by the analysis of [36]. It would be interesting to check this, as well as the details of generalizations to other dimensions. In particular, for  $d \neq 4$  it remains to determine the form of any associated boundary superfields. We note that a generalization to  $d = 5$  would address a number of interesting examples already well-studied in the context of AdS/CFT [26, 27, 28].

Let us return, however, to  $\mathcal{N} = 4$  supergravity in  $d = 4$ . Even here, one expects to obtain an additional class of  $\mathcal{N} = 1$  theories by deforming the boundary conditions for the vector super-multiplet. Such a possibility was indicated in the original work of Breitenlohner and Freedman [17], and in the context of AdS/CFT would amount to a supersymmetric generalization of [54, 33, 34, 55, 56, 35]. We have not considered this option here, but again it would be of interest to work out the full formalism and to obtain a useful form of any associated vector “boundary superfields.”

Finally, we have considered only boundary conditions for the graviton supermultiplet which impose the usual boundary condition on the graviton itself: namely, that the induced metric on the boundary is fixed. However, for  $d = 4$  one may also impose boundary conditions [18] which have slower fall-off near  $\mathcal{I}$ . In this case, AdS/CFT arguments [33, 34, 35] suggest that such boundary conditions are related to “dual” boundary theories which *also* contain gravity and which are defined on spacetimes with compact Cauchy surfaces. As a result, one expects that with such slower fall-off boundary conditions, all asymptotic symmetries will be gauge and lead only to trivial conserved charges. It would, however, be interesting to check that supersymmetry can be preserved in such contexts, and to check that indeed the associated variations  $\delta\mathcal{H}_\eta, \delta\mathcal{H}_X$  vanish on-shell.

## Acknowledgements

The authors would like to thank Rob Leigh for useful discussions. D.M. was supported in part by NSF grant PHY0354978, by funds from the University of California. S.H. would like to thank the physics department at UC Santa Barbara for their kind hospitality.

## A General properties of $\mathcal{H}_\eta$

In this appendix, we summarize some important, model independent general properties of the generators  $\mathcal{H}_\eta$  for any (bosonic or fermionic) symmetry  $\eta$ .

### A.1 Ambiguity of $\mathcal{H}_\eta$ :

Two potential ambiguities arise in the construction of  $\mathcal{H}_\eta$ . First, a potential ambiguity arises from the fact that we may change the Lagrangian  $\mathbf{L}$  to  $\mathbf{L} + d\mu$ , with  $\mu$  a  $(d-1)$ -form that is locally constructed out of the fields, without changing the field equations. However, it is easy to see that such a change will not affect the symplectic form  $\omega$ , and hence will not change the definition  $\mathcal{H}_\eta$ . Another potential ambiguity arises because  $\theta$  is only unique up to the addition of an exact form,

$$\theta(\Phi, \delta\Phi) \rightarrow \theta(\Phi, \delta\Phi) + d\mathbf{Z}(\Phi, \delta\Phi). \quad (\text{A.168})$$

The addition of such a term will change the definition of  $\mathcal{H}_\eta$ —assuming it exists—by

$$\mathcal{H}_\eta \rightarrow \mathcal{H}_\eta + \int_{\partial\Sigma} \mathbf{Z}(\Phi, s_\eta\Phi). \quad (\text{A.169})$$

This type of ambiguity will typically be eliminated by the given choice of boundary conditions, which ensure that the only  $\mathbf{Z}$ s which gives rise to finite modification of the generator  $\mathcal{H}_\eta$  simply shift  $\mathcal{H}_\eta$  by constants, if they exist at all.

### A.2 Gauge invariance of $\mathcal{H}_\eta$

Note that  $\mathbf{Q}_\eta$  and  $\mathbf{A}_\eta$  in the definition of  $\delta\mathcal{H}_\eta$  are by construction “potentials” for gauge invariant quantities, and therefore need not be gauge invariant themselves. Consequently  $\mathcal{H}_\eta$  itself is in danger of not being gauge invariant. We claim, however, that  $\delta\mathcal{H}_\eta$ , and hence  $\mathcal{H}_\eta$  itself, is always invariant under gauge transformations that can be continuously connected to the identity (i.e., except possibly for “large gauge transformations”). As an example, consider the case when one of the matter fields is a non-abelian gauge field  $A_\mu$ . If we assume that the Lagrangian is invariant under infinitesimal gauge transformations  $\delta A_\mu = \nabla_\mu\Lambda + [A_\mu, \Lambda]$ , then it follows (by going through the definitions), that  $\mathbf{Q}_\eta$  is invariant under such a transformation up to the addition of a closed form  $\mathbf{G}_\Lambda$ . However, again by the fundamental lemma, it follows that  $\mathbf{G}_\Lambda$  must be exact, and hence cannot contribute to the integral defining  $\delta\mathcal{H}_\xi$ . A similar argument can be made for the second term contributing to this integral. Hence,  $\delta\mathcal{H}_\eta$ , and therefore  $\mathcal{H}_\eta$  itself, is independent under infinitesimal gauge transformations. But this implies that it is independent under any finite gauge transformation that is connected to the identity by a differentiable path.

### A.3 Off-shell formula for $\mathcal{H}_\eta$

It is often useful to have an off-shell formula for  $\mathcal{H}_\eta$ , for example, for the computation of Poisson brackets with interesting gauge dependent quantities. To construct such a formula, we now compute  $\delta\mathcal{H}_\eta(\Phi)$  for any solution  $\Phi$ , but arbitrary off shell variations  $\delta\Phi$ . Note that, in this situation, our derivation of the variation of the generator  $\mathcal{H}_\eta$  only implies that

$$\delta\mathcal{H}_\eta = \int_\Sigma \delta\mathbf{J}_\eta + \int_{\partial\Sigma} \mathbf{A}_\eta. \quad (\text{A.170})$$

On shell, we have  $\mathbf{J}_\eta = d\mathbf{Q}_\eta$ , so we also have  $\delta\mathbf{J}_\eta = d\delta\mathbf{Q}_\eta$  for on shell variations  $\delta\Phi$ , and the first integral reduces to a surface integral  $\int \delta\mathbf{Q}_\eta$ . However, this is not true for off-shell variations, and the off-shell variation of  $\mathcal{H}_\eta$  consequently differs from (A.170) by a term that is not a boundary integral. To characterize this term, it is necessary to have a suitable off-shell formula for  $\mathbf{Q}_\eta$ . Since  $\mathbf{J}_\eta = d\mathbf{Q}_\eta$  only holds on shell, we have in general

$$(\mathbf{J}_\eta - d\mathbf{Q}_\eta)_{\nu_1 \dots \nu_{d-1}} = \sum_{I=0}^N \lambda^{I(\mu_1 \dots \mu_I)}{}_{A[\nu_1 \dots \nu_{d-1}]}(\Phi) \nabla_{(\mu_1} \dots \nabla_{\mu_I)} \eta^A, \quad (\text{A.171})$$

where  $\lambda^I(\Phi) = 0$  vanishes if  $\Phi$  is on shell. We will now show that, if the variation  $s_\eta\Phi$  contains at most  $M$  derivatives of  $\eta^A$ , then the Noether charge can be modified  $\mathbf{Q}_\eta \rightarrow \mathbf{Q}_\eta + \boldsymbol{\tau}_\eta$  by a  $(d-2)$ -form  $\boldsymbol{\tau}$  vanishing on shell, such that the right side of eq. (A.171) contains at most  $M-1$  derivatives of  $\eta^A$ . The proof of this statement is similar in nature to arguments given in [32]: Let the maximum number  $N$  of derivatives in eq. (A.171) be such that  $N \geq M$ . Since

$$d(\mathbf{J}_\eta - d\mathbf{Q}_\eta) = \mathbf{E} s_\eta\Phi, \quad (\text{A.172})$$

and since the right side of that equation by assumption contains at most  $M$  derivatives of  $\eta^A$ , it follows that

$$0 = \lambda^{(\mu_1 \dots \mu_N)}{}_{A[\nu_1 \dots \nu_{d-1}]} \delta^{(\sigma)}{}_{\rho]} \nabla_{(\mu_1} \dots \nabla_{\mu_N} \nabla_{\sigma)} \eta^A \quad (\text{A.173})$$

for all  $\eta^A$ . Consequently, we have

$$0 = \lambda^{(\mu_1 \dots \mu_N)}{}_{A[\nu_1 \dots \nu_{d-1}]} \delta^{(\sigma)}{}_{\rho]}. \quad (\text{A.174})$$

Defining now

$$\tau_{\nu_1 \dots \nu_{d-2}} = \frac{N}{N+1} \lambda^{\sigma\mu_1 \dots \mu_{N-1}}{}_{A\sigma[\nu_1 \dots \nu_{d-2}]} \nabla_{(\mu_1} \dots \nabla_{\mu_{N-1})} \eta^A, \quad (\text{A.175})$$

it may be checked that  $\mathbf{J}_\eta - d\mathbf{Q}_\eta - d\boldsymbol{\tau}_\eta$  contains at most  $N-1$  derivatives of  $\eta^A$ , and clearly  $\boldsymbol{\tau}(\Phi) = 0$  on shell. Therefore, the highest derivative term on the right side

of eq. (A.171) has been removed by an off-shell redefinition of  $\mathbf{Q}_\eta \rightarrow \mathbf{Q}_\eta - \boldsymbol{\tau}_\eta$  which does not affect the form of  $\mathbf{Q}_\eta$  on shell. We can continue this process inductively to remove the next-highest derivative term on the right side of (A.171), as long as the number of derivatives is not smaller than  $M$ . Therefore, by a suitable off-shell redefinition of  $\mathbf{Q}_\eta$ , it can always be achieved that

$$\mathbf{J}_\eta - d\mathbf{Q}_\eta = \sum_{I=0}^{M-1} \mathbf{C}_A^{\mu_1 \dots \mu_I}(\Phi) \nabla_{(\mu_1} \dots \nabla_{\mu_I)} \eta^A, \quad (\text{A.176})$$

where  $\mathbf{C}^A(\Phi)$  vanishes on shell, and where  $M$  is the maximum number of derivatives of  $\eta^A$  appearing in the symmetry variation  $s_\eta \Phi$ . Therefore, by eq. (A.170), the total off-shell Hamiltonian variation is given by

$$\delta \mathcal{H}_\eta = \sum_{I=0}^{M-1} \int_\Sigma \delta \left[ \mathbf{C}_A^{\mu_1 \dots \mu_I} \nabla_{(\mu_1} \dots \nabla_{\mu_I)} \eta^A \right] + \int_{\partial \Sigma} \delta \mathbf{Q}_\eta - \mathbf{A}_\eta. \quad (\text{A.177})$$

The forms  $\mathbf{C}_A^{\mu_1 \dots \mu_I}$  are the ‘‘constraints’’ associated with the fermionic symmetries, and vanish when the equations of motion hold. Thus, we see that the off shell variation of  $\mathcal{H}_\eta$  is not given by a boundary integral, but contains also a ‘‘bulk’’ integral containing the variation of the constraints of the theory associated with the fermionic symmetries. In particular, if the number of derivatives appearing in the SUSY transformations is  $M = 1$ , as happens e.g. when the Lagrangian contains no more than second derivatives of the fields (the case in nearly all applications, and in particular in the supergravities studied in this paper), then

$$\mathbf{J}_\eta - d\mathbf{Q}_\eta = \mathbf{C}_A \eta^A, \quad (\text{A.178})$$

and we get a correspondingly simpler form for the off shell variation of  $\mathcal{H}_\eta$ ,

$$\delta \mathcal{H}_\eta = \int_\Sigma \delta(\mathbf{C}_A \eta^A) + \int_{\partial \Sigma} \delta \mathbf{Q}_\eta - \mathbf{A}_\eta. \quad (\text{A.179})$$

## References

- [1] J. Lee and R. M. Wald, ‘‘Local symmetries and constraints,’’ *J. Math. Phys.* **31**, 725 (1990).
- [2] V. Iyer and R. M. Wald, ‘‘Some properties of Noether charge and a proposal for dynamical black hole entropy,’’ *Phys. Rev. D* **50**, 846 (1994) [arXiv:gr-qc/9403028].

- [3] V. Iyer and R. M. Wald, “A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes,” *Phys. Rev. D* **52**, 4430 (1995)
- [4] R. M. Wald and A. Zoupas, “A General Definition of ”Conserved Quantities” in General Relativity and Other Theories of Gravity,” *Phys. Rev. D* **61**, 084027 (2000)
- [5] A. Ashtekar and M. Streubel, “Symplectic Geometry Of Radiative Modes And Conserved Quantities At Null Infinity,” *Proc. Roy. Soc. Lond. A* **376**, 585 (1981),
- [6] A. Ashtekar, L. Bombelli and O. Reula, “The covariant phase space of asymptotically flat gravitational fields,” *Print-90-0318* (Syracuse), [http://www.slac.stanford.edu/spires/find/hep/www?r=print-90-0318%2F\(syracuse\)](http://www.slac.stanford.edu/spires/find/hep/www?r=print-90-0318%2F(syracuse))
- [7] P. T. Chrusciel, J. Jezierski, J. Kijowski: “Hamiltonian field theory in the radiating regime,” *Springer Lecture Notes in Physics* (2002)
- [8] S. Hollands and A. Ishibashi, “Asymptotic flatness and Bondi energy in higher dimensional gravity,” *J. Math. Phys.* **46**, 022503 (2005)
- [9] S. Hollands, A. Ishibashi and D. Marolf, “Comparison between various notions of conserved charges in asymptotically AdS-spacetimes,” *Class. Quant. Grav.* **22**, 2881 (2005)
- [10] I. Papadimitriou and K. Skenderis, “AdS/CFT correspondence and geometry,” *arXiv:hep-th/0404176*.
- [11] T. Hertog and S. Hollands, “Stability in designer gravity,” *Class. Quant. Grav.* **22**, 5323 (2005)
- [12] A. J. Amsel and D. Marolf, “Energy bounds in designer gravity,” *Phys. Rev. D* **74**, 064006 (2006)
- [13] P. K. Townsend, “Cosmological Constant In Supergravity,” *Phys. Rev. D* **15**, 2802 (1977).
- [14] M. Henneaux and C. Teitelboim, “Asymptotically Anti-De Sitter Spaces,” *Commun. Math. Phys.* **98**, 391 (1985).
- [15] D. Z. Freedman and J. H. Schwarz, “N=4 Supergravity Theory With Local SU(2) X SU(2) Invariance,” *Nucl. Phys. B* **137** (1978) 333.
- [16] A. Das, M. Fischler and M. Rocek: “Super Higgs-effect in a new class of scalar models and a model of super QED,” *Phys. Rev.* **D16** 3427 (1977)

- [17] P. Breitenlohner and D. Z. Freedman, “Stability In Gauged Extended Supergravity,” *Annals Phys.* **144** (1982) 249;
- [18] A. Ishibashi and R. M. Wald, “Dynamics in non-globally hyperbolic static spacetimes. III: anti-de Sitter spacetime,” *Class. Quant. Grav.* **21**, 2981 (2004) [arXiv:hep-th/0402184].
- [19] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2**, 231 (1998) [*Int. J. Theor. Phys.* **38**, 1113 (1999)]
- [20] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” *Phys. Lett. B* **428**, 105 (1998)
- [21] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2**, 253 (1998)
- [22] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” *Phys. Rept.* **323**, 183 (2000)
- [23] E. Witten, “Multi-trace operators, boundary conditions, and AdS/CFT correspondence,” arXiv:hep-th/0112258.
- [24] M. Berkooz, A. Sever and A. Shomer, “Double-trace deformations, boundary conditions and spacetime singularities,” *JHEP* **0205**, 034 (2002) [arXiv:hep-th/0112264].
- [25] A. Sever and A. Shomer, “A note on multi-trace deformations and AdS/CFT,” *JHEP* **0207**, 027 (2002) [arXiv:hep-th/0203168].
- [26] J. Polchinski and M. J. Strassler, “The string dual of a confining four-dimensional gauge theory,” arXiv:hep-th/0003136.
- [27] K. Pilch and N. P. Warner, “N = 1 supersymmetric renormalization group flows from IIB supergravity,” *Adv. Theor. Math. Phys.* **4**, 627 (2002) [arXiv:hep-th/0006066].
- [28] O. Lunin and J. M. Maldacena, “Deforming field theories with U(1) x U(1) global symmetry and their gravity duals,” *JHEP* **0505**, 033 (2005) [arXiv:hep-th/0502086].
- [29] Y. Choquet-Bruhat, “The Cauchy Problem In Classical Supergravity,” *Lett. Math. Phys.* **7**, 459 (1983).

- [30] D. Bao, Y. Choquet-Bruhat, J. Isenberg and P. B. Yasskin, “The Well Posedness Of (N=1) Classical Supergravity,” *J. Math. Phys.* **26**, 329 (1985).
- [31] D. Bao, “A Sufficient Condition For The Linearization Stability Of N=1 Supergravity: A Preliminary Report,” *Annals Phys.* **158**, 211 (1984).
- [32] R. M. Wald: “On identically closed forms locally constructed from a field,” *J. Math. Phys.* **31**, 2378 (1990)
- [33] R. G. Leigh and A. C. Petkou, “Holography of the N = 1 higher-spin theory on AdS(4),” *JHEP* **0306**, 011 (2003) [arXiv:hep-th/0304217];
- [34] “SL(2,Z) action on three-dimensional CFTs and holography,” *JHEP* **0312**, 020 (2003) [arXiv:hep-th/0309177].
- [35] D. Marolf and S. F. Ross, “Boundary conditions and dualities: Vector fields in AdS/CFT,” arXiv:hep-th/0606113.
- [36] P. van Nieuwenhuizen and D. V. Vassilevich, *Class. Quant. Grav.* **22**, 5029 (2005) [arXiv:hep-th/0507172].
- [37] C. Fefferman and C.R. Graham: “Conformal invariants,” in: *Elie Cartan et les Mathematiques d’Aujourd’hui*, Asterisque, 1985, Numero hors serie, Soc. Math. France, Paris, pp. 95-116.
- [38] A. A. Starobinsky: “Isotropization of arbitrary cosmological expansion given an effective cosmological constant,” *JETP Lett.* **37**, 66 (1983)
- [39] K. Skenderis, “Lecture notes on holographic renormalization,” *Class. Quant. Grav.* **19**, 5849 (2002) [arXiv:hep-th/0209067].
- [40] S. de Haro, S. N. Solodukhin and K. Skenderis, “Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence,” *Commun. Math. Phys.* **217**, 595 (2001) [arXiv:hep-th/0002230].
- [41] B. de Wit and H. Nicolai, “Extended Supergravity With Local SO(5) Invariance,” *Nucl. Phys. B* **188**, 98 (1981).
- [42] B. de Wit and H. Nicolai, “N=8 Supergravity With Local SO(8) X SU(8) Invariance,” *Phys. Lett. B* **108**, 285 (1982).
- [43] E. Cremmer and B. Julia, “The N=8 Supergravity Theory. 1. The Lagrangian,” *Phys. Lett. B* **80**, 48 (1978). E. Cremmer, B. Julia and J. Scherk, “Supergravity theory in 11 dimensions,” *Phys. Lett. B* **76**, 409 (1978).

- [44] W. Muck and K. S. Viswanathan, “The Wess-Zumino model and the AdS(4)/CFT(3) correspondence,” *Phys. Rev. D* **60**, 046003 (1999) [arXiv:hep-th/9903194].
- [45] S. W. Hawking, “The Boundary Conditions For Gauged Supergravity,” *Phys. Lett. B* **126**, 175 (1983).
- [46] P. K. Townsend, “Positive Energy And The Scalar Potential In Higher Dimensional (Super)Gravity Theories,” *Phys. Lett. B* **148**, 55 (1984).
- [47] W. Boucher, “Positive Energy without supersymmetry,” *Nucl. Phys.* **B242** (1984), 282
- [48] G. W. Gibbons, S. W. Hawking, G. T. Horowitz and M. J. Perry, “Positive Mass Theorems For Black Holes,” *Commun. Math. Phys.* **88**, 295 (1983).
- [49] G. W. Gibbons, C. M. Hull and N. P. Warner, “The Stability Of Gauged Supergravity,” *Nucl. Phys. B* **218**, 173 (1983).
- [50] J. A. Nester, “A New Gravitational Energy Expression With A Simple Positivity Proof,” *Phys. Lett. A* **83**, 241 (1981).
- [51] E. Witten, “A Simple Proof Of The Positive Energy Theorem,” *Commun. Math. Phys.* **80**, 381 (1981).
- [52] A. Amsel, T. Hertog, S. Hollands, D. Marolf: “A tale of two superpotentials, stability and instability in designer gravity,” to appear
- [53] T. Hertog, “Violation of energy bounds in designer gravity,” arXiv:hep-th/0607171.
- [54] E. Witten, “SL(2,Z) action on three-dimensional conformal field theories with Abelian symmetry,” arXiv:hep-th/0307041.
- [55] A. C. Petkou, “Holography, duality and higher-spin theories,” arXiv:hep-th/0410116.
- [56] H. U. Yee, “A note on AdS/CFT dual of SL(2,Z) action on 3D conformal field theories with U(1) symmetry,” *Phys. Lett. B* **598**, 139 (2004) [arXiv:hep-th/0402115].