

Comparison between various notions of conserved charges in asymptotically AdS-spacetimes

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Abstract

We derive hamiltonian generators of asymptotic symmetries for general relativity with asymptotic AdS boundary conditions using the “covariant phase space” method of Wald et al. We then compare our results with other definitions that have been proposed in the literature. We find that our definition agrees with that proposed by Ashtekar et al, with the spinor definition, and with the background dependent definition of Henneaux and Teitelboim. Our definition disagrees with the one obtained from the “counterterm subtraction method,” but the

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difference is found to consist only of a “constant offset” that is determined entirely in terms of the boundary metric. We finally discuss and justify our boundary conditions by a linear perturbation analysis, and we comment on generalizations of our boundary conditions, as well as inclusion of matter fields.

1 Introduction

While the cosmological constant observed in nature seems to be (small and) positive, there is considerable theoretical interest in studying theories of gravity with a negative cosmological constant. A negative cosmological constant gives rise to classical spacetimes whose asymptotic structure can be viewed as a timelike “boundary” at infinity, which in turn can be considered as a lower dimensional spacetime in its own right. The AdS/CFT correspondence [1, 2, 3, 4] asserts that conformal quantum field theories living on this boundary provide a “holographic” description of the gravity theory in the “bulk”.

An important aspect of the correspondence is the matching of symmetries on both sides. In diffeomorphism invariant theories such as general relativity in d dimensions, the hamiltonian generators of asymptotic spacetime symmetries can be expressed as surface integrals over a $(d-2)$ -dimensional cross section at infinity, while the asymptotic symmetry vector fields themselves can be identified with conformal killing fields of the boundary. On the field theory side, these generators correspond to generators of the conformal symmetry algebra of the CFT, which are also given by integrals over $(d-2)$ dimensional surfaces within the boundary spacetime¹.

Given the importance of symmetries for the correspondence, it is perhaps surprising that our understanding of the hamiltonian generators of asymptotic symmetries in asymptotically AdS spacetimes does not seem to be optimal—as may be appreciated e.g. from that fact that, over the years, several definitions for “conserved charges” associated with asymptotic symmetries have been proposed: The definition by Ashtekar et al. [5, 6] based on the electric Weyl tensor, the Hamiltonian definition by Henneaux and Teitelboim [7], the “pseudotensor” approach of Abbott and Deser [8], the

¹The same statement is also true for the generators of the “fermionic” symmetries that occur in supergravity. We will restrict ourselves here to the bosonic generators.

KBL approach [9, 10], the spinor definition [11, 12] based on an original idea by Witten [13], and the “counterterm subtraction method” by Henningson and Skenderis and by Balasubramanian and Kraus [14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. These constructions are all rather different in philosophy and appearance. Their relation has mostly been analyzed in the context of special solutions to the field equations, but a comprehensive and systematic comparison does not seem to exist in the literature. The aim of the present paper is to fill in this gap.

However, before doing so, we will begin in sections 2 and 3 by providing yet another construction of conserved charges in asymptotically AdS-spacetimes. For this purpose we use the general “covariant phase space formalism” of Wald et al. [24, 25] (see also [26] for earlier related work). Our construction combines, in some sense, many individual advantages of the previous constructions, while avoiding some of their disadvantages. For example, it has the advantage of being couched in a Hamiltonian framework (as does the approach of Henneaux and Teitelboim), which ensures that the charges have the proper physical interpretation as the generators of the symmetries. At the same time, the construction is manifestly “covariant” and “background independent” because it uses the formalism of conformal infinity introduced by Penrose [27] (as in Ashtekar’s definition). Our construction will also facilitate the comparison with various previous definitions of conserved charges.

This comparison is carried out in detail in section 4. It turns out that the final form of our expression for the conserved charges is manifestly equivalent to Ashtekar’s (although our derivation is different). In subsection 4.1, we compare our definition with the counterterm subtraction method in $d = 5$. We show that the methods are not equivalent, as had in fact been noticed previously [15, 6]. However, we show that the difference is given only by a constant offset, which is expressible entirely in terms of the non-dynamical boundary data, and hence is the same for *any* asymptotically AdS space-time. In subsection 4.2, we compare our method to that of Henneaux and Teitelboim, and we find that both are equivalent. In subsection 4.3, we establish that our definition is equivalent to the spinor method in d dimensions, thereby generalizing a result of Davis [28] in $d = 4$. However, comparison with the Abbott-Deser approach [8] and its generalizations [29, 30, 31], as well as with the KBL approach [9, 10], is postponed for future investigation. In section 5, we discuss how to generalize our constructions when matter fields

are present. Of particular interest are scalar fields saturating the so-called “Breitenlohner-Friedman bound” [32], and abelian p -form fields. These are therefore discussed in some detail in sections 5.1 and 5.2–5.4, respectively.

Finally, in section 6 we motivate our choice of “boundary conditions” by showing that they arise naturally in a linear perturbation analysis around exact AdS-space. In fact, that analysis shows that other boundary conditions are also possible, and we discuss some of these, although we leave a more detailed analysis, as well as the derivation of the corresponding conserved charges to a future investigation.

Our **notations and conventions** are the following: The dimension of the spacetime is denoted d , and we assume² $d \geq 4$. The signature of the metric is $(-+++ \dots)$, the convention for the Riemann tensor is $\nabla_{[a}\nabla_{b]}k_c = (1/2)R_{abc}{}^d k_d$ and $R_{ab} = R_{acb}{}^c$ for the Ricci tensor. Indices in parenthesis are symmetrized and indices in brackets are antisymmetrized. Indices on tilde tensor fields $\tilde{t}_{abc\dots}$ are raised and lowered with the unphysical metric \tilde{g}_{ab} and its inverse \tilde{g}^{ab} , whereas indices on untilde fields are raised and lowered with the physical metric $g_{ab} = \Omega^{-2}\tilde{g}_{ab}$, and its inverse g^{ab} . The AdS radius is denoted ℓ . We set $\ell = 1$ in most of our formulas if not explicitly stated otherwise.

2 Hamiltonian approach to conserved charges in AdS and boundary conditions

In this section, we review the general algorithm given by Wald and Zoupas [24] for defining “charges” associated with symmetries preserving a given set of “boundary conditions” in the context of theories derived from a diffeomorphism covariant Lagrangian. This will be used to define conserved charges in d -dimensional general relativity with a specific choice of asymptotic AdS

²Since the Weyl tensor vanishes for $d = 3$ the definition [5, 6] of Ashtekar et al becomes trivial for this case. However, for $d = 3$ a covariant phase space construction of energy has already been given in [33] and one has the Henneaux-Teitelboim type construction [34] by Brown and Henneaux. While it would be interesting to compare these latter definitions with the counter-term subtraction approach, the $d = 3$ case requires special treatment due to the appearance of factors of $d - 3$ in many formulae below. We therefore restrict to $d \geq 4$ below, referring the reader to [17] for details of the expansion in $d = 3$ and saving the comparison of charges for [35].

boundary conditions, as the generator conjugate to an appropriately defined asymptotic symmetry.

The algorithm [24] applies to arbitrary theories derived from a diffeomorphism covariant Lagrangian. We focus here on vacuum general relativity with a negative cosmological constant in d dimensions, defined by the Lagrangian density (viewed as a d -form)

$$L = \frac{1}{16\pi G} \sqrt{-g} (R - 2\Lambda) d^d x, \quad \Lambda < 0. \quad (1)$$

In order to completely specify the theory, one must also prescribe a set of asymptotic conditions for the metric. In this paper, we will consider the following

Asymptotic Conditions:

1. One can attach a boundary, $\mathcal{I} \cong R \times S^{d-2}$ to M such that $\tilde{M} = M \cup \mathcal{I}$ is a manifold with boundary.
2. On \tilde{M} , there is a smooth³ metric \tilde{g}_{ab} and a smooth function Ω such that $g_{ab} = \Omega^{-2} \tilde{g}_{ab}$, and such that $\Omega = 0$ and

$$\tilde{n}_a \equiv \tilde{\nabla}_a \Omega \neq 0 \quad (2)$$

at points of \mathcal{I} . The metric \tilde{h}_{ab} on \mathcal{I} induced by \tilde{g}_{ab} is in the conformal class of the Einstein static universe,

$$\tilde{h}_{ab} dx^a dx^b = e^\omega [-dt^2 + d\sigma^2], \quad (3)$$

where $d\sigma^2$ is the line element of the unit sphere S^{d-2} , and where ω is some smooth function. Thus, \mathcal{I} is a timelike boundary.

Other boundary conditions, corresponding to different notions of asymptotically AdS spacetimes are also possible (see Sec.6), but will not be discussed

³By “smooth”, we mean C^∞ . However, for our constructions to work, it would be sufficient to require only that \tilde{g}_{ab} is $(d-1)$ -times continuously differentiable. This weaker requirement is, in fact, the appropriate one when various matter fields are included. It would then be natural to also weaken the differentiability of Ω and the manifold structure of \tilde{M} , but we will not discuss this for simplicity.

here. The prototype spacetime satisfying the above asymptotic conditions is, of course, AdS space itself. It has the line element

$$ds_0^2 = -(1 + r^2/\ell^2) dt^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\sigma^2, \quad (4)$$

in global coordinates, where

$$\ell \equiv \sqrt{-(d-1)(d-2)/2\Lambda} \quad (5)$$

is the AdS radius. By a change of coordinates, this can be brought into the form

$$ds_0^2 = \frac{\ell^2}{\Omega^2} \left[d\Omega^2 - dt^2 + d\sigma^2 - \frac{\Omega^2}{2}(dt^2 + d\sigma^2) + \frac{\Omega^4}{16}(-dt^2 + d\sigma^2) \right], \quad (6)$$

where $\Omega > 0$ is an analytic function of r . Thus, for pure AdS, a conformal completion can be obtained by taking \tilde{M} to be the manifold obtained from M by attaching the boundary \mathcal{S} consisting of the points $\Omega = 0$, and by taking the unphysical metric to be $d\tilde{s}_0^2 = \Omega^2 ds_0^2$. This is explicitly

$$d\tilde{s}_0^2 = \ell^2 \left[d\Omega^2 - dt^2 + d\sigma^2 - \frac{\Omega^2}{2}(dt^2 + d\sigma^2) + \frac{\Omega^4}{16}(-dt^2 + d\sigma^2) \right], \quad (7)$$

and obviously smooth. The induced metric on \mathcal{S} is $\ell^2[-dt^2 + d\sigma^2]$, i.e., the metric of the Einstein static universe with radius ℓ . One can likewise verify that the asymptotic conditions are also obeyed by the AdS-Schwarzschild and AdS-Myers-Perry solutions.

If the above asymptotic conditions are combined with Einstein's equation, then one can obtain much more detailed results about the asymptotic form of the metric at infinity. These consequences will be worked out in the next section and are summarized in lemma 3.1.

The diffeomorphisms f of \tilde{M} with the property that f^*g_{ab} is asymptotically AdS whenever g_{ab} is, form a group under composition. Of physical significance is the group G obtained by factoring this group by $\text{Diff}(M)_0$, where $\text{Diff}(M)_0$ is the subgroup of diffeos leaving \mathcal{S} pointwise invariant. The factor group G is called the ‘‘asymptotic symmetry group’’. Since the elements of G can be identified with conformal isometries of the Einstein static universe, it follows that

$$G \cong O(d-1, 2). \quad (8)$$

The elements of the Lie-algebra of G correspond to equivalence classes of vector fields ξ^a generating a 1-parameter group asymptotic symmetries, modulo vector fields that vanish on \mathcal{S} . (By abuse of language, we will refer to these again as “asymptotic symmetries.”) We are interested in defining the corresponding generators \mathcal{H}_ξ on phase space⁴.

For this, consider the variation of the Lagrange density L , which can always be written in the form

$$\delta L = F \cdot \delta g + d\theta, \quad (9)$$

Here, F are the field equations; in our case

$$F_{ab} = \frac{1}{16\pi G} d^d x \sqrt{-g} \left(R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} \right); \quad (10)$$

and $d\theta$ is the exterior differential of the $(d-1)$ -form θ corresponding to the “boundary term” that would arise if the variation of L were performed under an integral sign. It is given in our case by

$$\theta_{a_1 \dots a_{d-1}} = \frac{1}{16\pi G} v^c \epsilon_{ca_1 \dots a_{d-1}}, \quad (11)$$

where $\epsilon = d^d x \sqrt{-g}$ is the volume form (identified with a d -form), and v_a is given by

$$v^a = \nabla^b \delta g_b^a - \nabla^a \delta g_b^b. \quad (12)$$

The antisymmetrized second variation⁵ ω of θ defines the (dualized) symplectic current,

$$\omega(g; \delta_1 g, \delta_2 g) = \delta_1 \theta(g; \delta_2 g) - \delta_2 \theta(g; \delta_1 g), \quad (13)$$

so that ω depends on the unperturbed metric and is skew in the pair of perturbations $(\delta_1 g, \delta_2 g)$. It is given in our case by

$$\omega_{a_1 \dots a_{d-1}} = \frac{1}{16\pi G} w^c \epsilon_{ca_1 \dots a_{d-1}}, \quad (14)$$

⁴Note that, since the Lie-algebra elements of G are in correspondence with equivalence classes vector fields on \tilde{M} modulo vector fields that vanish on \mathcal{S} , it follows that the expression for \mathcal{H}_ξ must be such that it is independent on which particular vector field representative in the equivalence class is chosen. Thus, it must vanish for any vector field that is zero on \mathcal{S} . This means, roughly speaking, that \mathcal{H}_ξ cannot depend on derivatives of ξ normal to \mathcal{S} .

⁵Here, and in similar other formulas involving second variations, we assume without loss of generality that the variations commute, i.e., that $\delta_1(\delta_2 g) - \delta_2(\delta_1 g) = 0$.

where w^c is the symplectic current vector

$$w^a = P^{abcdef}(\delta_1 g_{bc} \nabla_d \delta_2 g_{ef} - \delta_2 g_{bc} \nabla_d \delta_1 g_{ef}) \quad (15)$$

with

$$P^{abcdef} = g^{ae} g^{fb} g^{cd} - \frac{1}{2} g^{ad} g^{be} g^{fc} - \frac{1}{2} g^{ab} g^{cd} g^{ef} - \frac{1}{2} g^{bc} g^{ae} g^{fd} + \frac{1}{2} g^{bc} g^{ad} g^{ef}. \quad (16)$$

The integral of the symplectic current over an achronal $(d-1)$ -dimensional submanifold Σ of \tilde{M} defines the symplectic structure, σ_Σ , of general relativity

$$\sigma_\Sigma(g; \delta_1 g, \delta_2 g) = \int_\Sigma \omega(g; \delta_1 g, \delta_2 g). \quad (17)$$

It follows from a general argument that when both $\delta_1 g$ and $\delta_2 g$ satisfy the linearized equations of motion, then $d\omega = 0$, or, what is the same thing, that the symplectic current (15) is conserved, $\nabla^a w_a = 0$. This fact can be used to find how σ_Σ depends upon the choice of Σ . Namely, let Σ_1 and Σ_2 be two achronal surfaces ending on \mathcal{I} . They enclose a spacetime volume which is bounded by Σ_1, Σ_2 , and a portion \mathcal{I}_{12} of infinity. By Stokes theorem, the difference between σ_1 and σ_2 is given by the integral $\int_{\mathcal{I}_{12}} \omega$. However, below equation (67) we will show that, when viewed as a form on the unphysical spacetime, the pull-back of ω to \mathcal{I} vanishes for linearized solutions $\delta_1 g$ and $\delta_2 g$ satisfying our asymptotic conditions. Consequently, the symplectic structure σ_Σ does not depend on the choice of Σ . Furthermore, since ω vanishes at \mathcal{I} and Σ is compact in the the unphysical spacetime, this will show that the symplectic structure (17) is well-defined. An equivalent—and somewhat more familiar—way to write the symplectic form is

$$\sigma_\Sigma(g; \delta_1 g, \delta_2 g) = \frac{1}{16\pi G} \int_\Sigma [\delta_1 \pi^{ij} \delta_2 q_{ij} - (1 \leftrightarrow 2)] \quad (18)$$

where q_{ij} is the intrinsic metric on Σ , and $\pi^{ij} = d^{d-1} x \sqrt{-q} (\kappa^{ij} - \kappa q^{ij})$ the momentum, written in terms of the extrinsic curvature κ_{ij} of Σ .

One would now like to define the generator associated with a vector field ξ^a representing an asymptotic symmetry by

$$\delta \mathcal{H}_\xi = \sigma_\Sigma(g; \delta g, \mathcal{L}_\xi g) \quad \forall \delta g, \quad (19)$$

where Σ is a partial Cauchy surface whose boundary, C , is a cut of \mathcal{S} . Note that, since the right side is the symplectic form, this equation says that \mathcal{H}_ξ —if it exists—is indeed the generator (in the sense of Hamiltonian mechanics) of the infinitesimal displacement (“Hamiltonian vector field”) $\delta g = \mathcal{L}_\xi g$, which in turn describes the action of an infinitesimal symmetry in the phase space of the theory.

Note that since $\mathcal{L}_\xi \mathcal{L}_\eta - \mathcal{L}_\eta \mathcal{L}_\xi = \mathcal{L}_{[\xi, \eta]}$ (with $[\xi, \eta]^a = \xi^b \nabla_b \eta^a - \eta^b \nabla_b \xi^a$ the commutator of two vector fields), it follows automatically that the Hamiltonian vector fields associated with two infinitesimal symmetries ξ^a, η^a satisfy the same algebra as ordinary vector fields on M under the commutator. Consequently, if the symplectic form σ_Σ is used to define a Poisson bracket, then the charges \mathcal{H}_ξ —if they exist—must satisfy the same algebra up to a central extension, i.e.,

$$\{\mathcal{H}_\xi, \mathcal{H}_\eta\} = \mathcal{H}_{[\xi, \eta]} + c(\xi, \eta). \quad (20)$$

Note also that at this stage $\mathcal{H}_\xi, \mathcal{H}_\eta$, and $c(\xi, \eta)$ might perhaps depend on the choice of partial Cauchy surface Σ .

Before analyzing the existence of \mathcal{H}_ξ within the specific setup under investigation in this paper—namely Einstein gravity with a negative cosmological constant and the boundary conditions spelled out above—it is instructive, following [24], to first study the general structure of eq. (19). Let us assume that the the right side of eq. (19) is actually finite, as will be the case, e.g., if the symplectic current form $\omega(g, \delta_1 g, \delta_2 g)$ has a well-defined (i.e., finite) extension to \mathcal{S} , and as will turn out to be the case in our setup. Equation (19) can then be written in the form [24]

$$\delta \mathcal{H}_\xi = \int_\Sigma \delta \mathcal{C}_a \xi^a + \int_C [\delta Q_\xi - \xi \cdot \theta(g, \delta g)], \quad (21)$$

Here, $\mathcal{C}_a = \mathcal{C}_{a[b_1 \dots b_{d-1}]}$ are the constraints of the theory⁶ (identified with $(d-1)$ -forms), and Q_ξ is the Noether charge, which is given in our case by

$$Q_{a_1 \dots a_{d-2}} = -\frac{1}{16\pi G} (\nabla^b \xi^c) \epsilon_{bca_1 \dots a_{d-2}}. \quad (22)$$

Consistency requires $(\delta_1 \delta_2 - \delta_2 \delta_1) \mathcal{H}_\xi = 0$, so we must have

$$0 = \xi \cdot [\delta_2 \theta(g, \delta_1 g) - \delta_1 \theta(g, \delta_2 g)] = \xi \cdot \omega(g; \delta_1 g, \delta_2 g) \quad (23)$$

⁶In the case of pure gravity that we are considering here, these are given by $\mathcal{C}_{a[b_1 \dots b_{d-1}]} = \epsilon_{b_1 \dots b_{d-1} c} (R_a{}^c - \frac{1}{2} R \delta_a{}^c + \Lambda \delta_a{}^c)$.

on \mathcal{I} , or else \mathcal{H}_ξ cannot exist. If this equation holds—which is by no means guaranteed in general and depends crucially on the Lagrangian *and* on the nature of the boundary conditions—then it follows⁷ that there is a $(d - 2)$ -form I_ξ such that

$$\delta Q_\xi - \xi \cdot \theta(\delta g) = \delta I_\xi \quad (24)$$

up to an exact form. We conclude that a solution to eq. (21) is given by

$$\mathcal{H}_\xi = \int_\Sigma \xi^a \mathcal{C}_a + \int_C I_\xi. \quad (25)$$

When the equations of motion are satisfied, then the constraints \mathcal{C}_a vanish identically, so the first term vanishes and \mathcal{H}_ξ reduces to a surface integral.

As we will see, the consistency condition (23) holds (in fact, the symplectic form ω vanishes identically on \mathcal{I}) under our asymptotic conditions (in combination with Einstein’s equations), and so a conserved generator \mathcal{H}_ξ exists. An explicit expression for this generator will be derived in the next section. Note that eq. (21) only fixes \mathcal{H}_ξ up to terms that have vanishing variation, i.e., terms that are defined entirely in terms of the background structure, which in our case is the geometry of the boundary.

It is natural to fix this non-uniqueness by requiring that $\mathcal{H}_\xi(g_0) = 0$ for all asymptotic symmetries in exact AdS. Now, note that given an exactly AdS metric g_0 we may take any asymptotic symmetry η to be represented by a Killing field of g_0 . As a result, the change $\delta_\eta \mathcal{H}_\xi$ may be evaluated by taking $\delta g = \mathcal{L}_\eta g_0 = 0$ in (19). Thus we find

$$\{\mathcal{H}_\eta, \mathcal{H}_\xi\}(g_0) = \delta_\eta \mathcal{H}_\xi(g_0) = 0. \quad (26)$$

As a result, taking $\mathcal{H}_\xi(g_0) = 0$ sets $c(\xi, \eta) = 0$ in (20) and ensures that our generators satisfy the symmetry algebra.

Furthermore, under the assumption that the symplectic structure σ_Σ is independent of Σ , taking $\mathcal{H}_\xi(g_0) = 0$ guarantees that \mathcal{H}_ξ is independent of the cut C . This result is manifest when \mathcal{H}_ξ is evaluated on g_0 . For a general asymptotically AdS-metric, we may use the fact that (19) is independent of Σ to establish that the failure (if any) of \mathcal{H}_ξ to be independent of the cut C is given by an expression that has vanishing variation. Assuming that any

⁷In order to prove this statement from the consistency condition (23), one needs to assume that the space of asymptotic AdS-geometries is simply connected [24].

asymptotic AdS-geometry can be connected to g_0 by a path, it follows that this expression in fact has to vanish. We will establish this explicitly for the theory of interest at the end of section 3 below.

It is worth contrasting the case of $\Lambda < 0$ with asymptotically AdS boundary conditions with the case $\Lambda = 0$ and asymptotically *flat* boundary conditions. In the latter case, it is found [36, 24] that eq. (23) is *not* satisfied and consequently, no “absolutely conserved” generator exists in that case. This is directly related to the fact that, in the asymptotically flat case, gravitational radiation can leak out to infinity. In [36] it is shown how to construct a canonical “non-conserved” generator $\mathcal{H}_\xi[C]$ (equal to the Bondi energy-momentum and angular momentum) satisfying the “balance law” $\mathcal{H}_\xi[C_1] - \mathcal{H}_\xi[C_2] = \int_{\mathcal{S}_{12}} F_\xi$, where F_ξ is a suitably defined flux-density⁸ through the portion \mathcal{S}_{12} of scri bounded by C_1 and C_2 .

3 AdS charges

We now implement the strategy laid out in the previous section. The upshot of our analysis will be that a conserved generator \mathcal{H}_ξ exists for every asymptotic symmetry ξ^a under our boundary conditions. It is given by

$$\mathcal{H}_\xi = \frac{-\ell}{8\pi G} \int_C \tilde{E}_{ab} \tilde{u}^a \xi^b d\tilde{S}. \quad (27)$$

where $d\tilde{S}$ is the integration element on C obtained from the unphysical metric, where \tilde{u}^a is the unit timelike normal to C within \mathcal{S} (normalized with respect to the unphysical metric), and where \tilde{E}_{ab} is the normalized (leading order) electric part of the unphysical Weyl tensor,

$$\tilde{E}_{ab} = \frac{1}{d-3} \Omega^{3-d} \tilde{C}_{abcd} \tilde{n}^c \tilde{n}^d. \quad (28)$$

The vector field \tilde{n}^a is defined as above by $\tilde{n}^a = \tilde{\nabla}^a \Omega$, and we remind the reader of our convention that tensor indices on “tilde” quantities are raised and lowered with the unphysical metric, while indices of “untilde” quantities are raised and lowered with the physical metric. It will be shown below that,

⁸When ξ^a is a time-translation, that flux is given by the square of a suitably defined “news tensor”.

despite the inverse powers of Ω , the quantity \tilde{E}_{ab} is finite at \mathcal{I} when the metric satisfies our boundary conditions and Einstein's equation.

Expression (27) agrees with the expression proposed previously by Ashtekar and Magnon [5] (in $d = 4$) and by Ashtekar and Das [6] (for higher dimensions). We emphasize, however, that our strategy leading to this expression is logically rather different from that of Ashtekar et al. First, we *derive* this expression within a Hamiltonian framework, whereas that expression was essentially guessed by Ashtekar et al., based on dimensional considerations, on the fact that it reproduces the known expressions for energy-momentum and angular momentum in the exactly known AdS-black hole spacetimes, and on the fact that \mathcal{H}_ξ turns out to be independent of the cross section C (see below). Secondly, while Ashtekar et al. essentially assume the finiteness of \tilde{E}_{ab} as part of their boundary conditions, we in fact derive the finiteness of \tilde{E}_{ab} .

As explained in the previous section, the fact that \mathcal{H}_ξ is derived from a consistent Hamiltonian framework together with the statement that \mathcal{H}_ξ vanishes when evaluated on exact AdS space automatically implies that it is conserved, i.e., does not depend on the cross section C . Actually, as pointed out by Ashtekar et al., this can also be seen explicitly. To see this, one notes that, by definition, \tilde{E}_{ab} is trace-free and symmetric. It will be shown below that furthermore

$$\tilde{D}^a \tilde{E}_{ab} = 0 \quad \text{on } \mathcal{I}, \quad (29)$$

where \tilde{D}_a is the intrinsic unphysical derivative operator on \mathcal{I} , i.e., the derivative operator of $e^\omega(-dt^2 + d\sigma^2)$. Now the difference between the hamiltonian charge for different cuts C_1, C_2 of \mathcal{I} can be written as

$$\mathcal{H}_\xi[C_1] - \mathcal{H}_\xi[C_2] = -\frac{-\ell}{8\pi G} \int_{\mathcal{I}_{12}} \tilde{D}^a (\tilde{E}_{ab} \xi^b) d\tilde{s}, \quad (30)$$

using Stoke's theorem, where $d\tilde{s}$ is now the $(d - 1)$ -dimensional integration element on \mathcal{I} associated with the unphysical metric. But it follows from the properties of \tilde{E}_{ab} that, on \mathcal{I} ,

$$\tilde{D}^b (\tilde{E}_{ab} \xi^a) = \tilde{E}^{ab} \tilde{D}_{(a} \tilde{\xi}_{b)} = \frac{1}{d-1} \tilde{E}_a{}^a \tilde{D}_b \xi^b = 0, \quad (31)$$

where we have used that an asymptotic symmetry restricts to a conformal Killing field on \mathcal{I} . Hence, the integrand of \mathcal{H}_ξ is divergence free, and we conclude that the integral does not depend on the choice of the cut C .

Let us now derive the expression for \mathcal{H}_ξ , and the properties of \tilde{E}_{ab} . As explained in the previous section, \mathcal{H}_ξ is uniquely defined by eq. (21) specifying its variation $\delta\mathcal{H}_\xi$, and by the property that $\mathcal{H}_\xi = 0$ in exact AdS space. The second property is obviously satisfied, because exact AdS space has a vanishing Weyl tensor, and hence a vanishing \tilde{E}_{ab} . In order to verify that our expression for \mathcal{H}_ξ has the correct variation postulated by eq. (21), we must analyze the consequences of Einstein's equations. The reader not interested in the details of this analysis can jump directly to lemma 3.1, where we summarize the results.

To analyze Einstein's equations, it is convenient to introduce the tensor field

$$\tilde{S}_{ab} = \frac{2}{d-2}\tilde{R}_{ab} - \frac{1}{(d-1)(d-2)}\tilde{R}\tilde{g}_{ab}, \quad (32)$$

i.e., \tilde{S}_{ab} is essentially the Ricci tensor of the unphysical metric \tilde{g}_{ab} . In terms of this field, Einstein's equation is

$$0 = \tilde{S}_{ab} + 2\Omega^{-1}\tilde{\nabla}_a\tilde{n}_b - \Omega^{-2}\tilde{g}_{ab}(\tilde{n}^c\tilde{n}_c - \ell^{-2}). \quad (33)$$

From now on, we set

$$\ell = 1. \quad (34)$$

Multiplying eq. (33) by Ω^2 and evaluating the result at \mathcal{S} , we see that

$$\tilde{n}^c\tilde{n}_c \upharpoonright \mathcal{S} = 1, \quad (35)$$

i.e., \tilde{n}^a is spacelike, unit, and normal to \mathcal{S} , consistent with our assumption that \mathcal{S} is a timelike boundary. Now it is always possible to choose the conformal factor Ω so that the unphysical metric takes the ‘‘Gaussian normal form’’

$$\tilde{g}_{ab} = \tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \tilde{h}_{ab}, \quad (36)$$

where $\tilde{h}_{ab} \equiv \tilde{h}_{ab}(\Omega)$ is such that $\tilde{h}_{ab}(\Omega = 0)$ is equal to the metric of the Einstein static universe on \mathcal{S} , and such that

$$\tilde{h}_{ab}\tilde{\nabla}^b\Omega = 0, \quad \tilde{g}^{ab}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega = 1 \quad (37)$$

in a full neighborhood of \mathcal{S} , (as usual, indices on tilda tensor fields are raised and lowered with \tilde{g}_{ab}). In other words, \tilde{h}_{ab} is the induced metric on the surfaces \mathcal{S}_Ω , the timelike surfaces of constant Ω . For example, for the

metric of exact AdS space, the choice of the conformal factor given by eq. (7) satisfies eqs. (37), and the form of \tilde{h}_{ab} can be read off from expression (7) as

$$\tilde{h}_{ab} dx^a dx^b = - \left(1 + \frac{1}{4}\Omega^2\right)^2 dt^2 + \left(1 - \frac{1}{4}\Omega^2\right)^2 d\sigma^2 \quad \text{in exact AdS.} \quad (38)$$

For a general asymptotically AdS metric, a conformal factor satisfying eq. (37) may be found by first choosing an arbitrary Ω , and then modifying this choice if necessary as $\Omega \rightarrow e^\omega \Omega$, where ω is to be determined. One way to do this is by making a (formal) power series ansatz $\omega = \sum \omega_i \Omega^i$, where $\mathcal{L}_{\tilde{n}} \omega_i = 0$. One chooses ω_0 so that the induced unphysical metric on \mathcal{S} is the Einstein static universe, and ω_1 so that $\tilde{\nabla}_a \tilde{n}_b = 0$ on \mathcal{S} . Einstein's equations (33) then immediately show that with these choices, we have $\tilde{n}^c \tilde{n}_c = 1 + O(\Omega^2)$. It can be seen from this that the remaining $\omega_2, \omega_3, \dots$ may then be chosen recursively so that $\tilde{n}_c \tilde{n}^c = 1$ for the new choice of conformal factor, to all orders in Ω . Consequently, we can achieve that eq. (37) holds to all orders in Ω , and that the induced metric on \mathcal{S} is the Einstein static universe. For the arguments given below, it will not matter whether eq. (37) holds exactly, or only to all orders in Ω . For simplicity, we will assume that it holds exactly.

For a conformal factor satisfying (37), Einstein's equation (33) simplifies to

$$\tilde{S}_{ab} = -2\Omega^{-1} \tilde{\nabla}_a \tilde{n}_b. \quad (39)$$

We now use the standard technique of splitting this equation into its components parallel and normal to \mathcal{S}_Ω (the surfaces of constant Ω) in a manner similar to the split performed in the ADM-formalism. If this is done, then the following set of ‘‘constraint’’ and ‘‘evolution’’ equations is obtained. The constraint equations are

$$-\tilde{\mathcal{R}} - \tilde{K}_{ab} \tilde{K}^{ab} + \tilde{K}^2 + 2(d-2)\Omega^{-1} \tilde{K} = 0, \quad (40)$$

$$\tilde{D}^a \tilde{K}_{ab} - \tilde{D}_b \tilde{K} = 0, \quad (41)$$

where \tilde{D}_a is the derivative operator associated with \tilde{h}_{ab} , $\tilde{K}_{ab} = -\tilde{h}_a{}^c \tilde{h}_b{}^d \tilde{\nabla}_c \tilde{n}_d$ is the extrinsic curvature of the surfaces \mathcal{S}_Ω (with respect to the unphysical metric), and where $\tilde{\mathcal{R}}_a{}^b$ is the intrinsic Ricci tensor. The evolution equations

are⁹

$$\frac{d}{d\Omega}\tilde{K}_a{}^b = -\tilde{\mathcal{R}}_a{}^b + \tilde{K}\tilde{K}_a{}^b + \Omega^{-1}(d-2)\tilde{K}_a{}^b + \Omega^{-1}\tilde{K}\delta_a{}^b, \quad (42)$$

$$\frac{d}{d\Omega}\tilde{h}_{ab} = -2\tilde{h}_{bc}\tilde{K}_a{}^c. \quad (43)$$

By assumption, \mathcal{S} is a smooth boundary, implying that the fields $\tilde{h}_{ab}, \tilde{K}_{ab}$ must be smooth in a neighborhood of \mathcal{S} . Consequently, multiplying the first evolution equation by Ω and evaluating on \mathcal{S} , we immediately conclude that

$$\tilde{K}_{ab} \upharpoonright \mathcal{S} = 0 = \frac{d}{d\Omega}\tilde{h}_{ab} \upharpoonright \mathcal{S}. \quad (44)$$

To investigate more systematically the consequences implied by eqs. (42, 43), we express them in terms of the traceless part $\tilde{p}_a{}^b$ of $\tilde{K}_a{}^b$ and use the familiar technique (see e.g. [37, 20, 38]) of performing the Taylor expansions:

$$\tilde{h}_{ab} = \sum_{j=0}^{\infty} \Omega^j (\tilde{h}_{ab})_j, \quad \tilde{p}_a{}^b = \sum_{j=0}^{\infty} \Omega^j (\tilde{p}_a{}^b)_j, \quad (45)$$

where each tensor $(\tilde{h}_{ab})_j, (\tilde{p}_a{}^b)_j$ is independent of Ω in the sense that the Lie-derivative along \tilde{n}^a vanishes. This yields the following recursion relations:

$$(d-2-j)(\tilde{p}_a{}^b)_j = (\tilde{\mathcal{R}}_a{}^b)_{j-1} - \frac{1}{d-1}(\tilde{\mathcal{R}})_{j-1}\delta_a{}^b - \sum_{m=0}^{j-1} (\tilde{K})_m (\tilde{p}_a{}^b)_{j-1-m}, \quad (46)$$

$$(2d-3-j)(\tilde{K})_j = (\tilde{\mathcal{R}})_{j-1} - \sum_{m=0}^{j-1} (\tilde{K})_m (\tilde{K})_{j-1-m}, \quad (47)$$

as well as

$$j(\tilde{h}_{ab})_j = -2 \sum_{m=0}^{j-1} \left[(\tilde{h}_{bc})_m (\tilde{p}_a{}^c)_{j-1-m} + \frac{1}{d-1} (\tilde{h}_{ab})_m (\tilde{K})_{j-1-m} \right], \quad (48)$$

⁹The symbols “ $d/d\Omega$ ” should be understood geometrically as the Lie derivative along \tilde{n}^a .

where we remind the reader that these equations hold for the choice (37) of the conformal factor. The “initial conditions” are, from eq. (44)

$$(\tilde{p}_a{}^b)_0 = (\tilde{K})_0 = 0, \quad (49)$$

and that $(\tilde{h}_{ab})_0 = -(dt)_a(dt)_b + \sigma_{ab}$ be the metric of the Einstein static universe. The key point about these equations is that $(\tilde{h}_{ab})_j$ and $(\tilde{K}_a{}^b)_l$ are uniquely determined in terms of the initial conditions for $j < d - 1$ and $l < d - 2$. Therefore, they must be equal to the corresponding quantities for exact AdS space, i.e., they are entirely “kinematical”. Thus, any quantity that depends only on $(\tilde{h}_{ab})_j$ and $(\tilde{K}_a{}^b)_l$ for j and l in this range, must automatically be equal to the corresponding quantity in pure AdS space. In particular, since the Weyl tensor vanishes identically in pure AdS space, it follows that

$$(\tilde{C}^a{}_{bcd})_j = 0 \quad \text{for } j < d - 3 \quad (50)$$

in *any* asymptotically AdS spacetime satisfying the Einstein equations. Furthermore, since \tilde{h}_{ab} for exact AdS space is given by equation (38) it follows that

$$\tilde{h}_{ab} dx^a dx^b = - \left(1 + \frac{1}{4}\Omega^2\right)^2 dt^2 + \left(1 - \frac{1}{4}\Omega^2\right)^2 d\sigma^2 + O(\Omega^{d-1}). \quad (51)$$

We will use these results later.

Let us now look at the recursion relation (46) for $j = d - 2$. The left side of this equation is given by $0 \cdot (\tilde{p}_a{}^b)_{d-2}$, so it does not yield any restriction on this coefficient¹⁰. The only restriction on this term comes from the constraint equation, which fixes the divergence of $(\tilde{p}_a{}^b)_{d-2}$ with respect to the derivative operator associated with the boundary metric. It can be seen that, once a traceless, symmetric tensor $(\tilde{p}_a{}^b)_{d-2}$ with the prescribed divergence is given, all tensors $(\tilde{p}_a{}^b)_j$, $(\tilde{h}_{ab})_j$ are uniquely determined for $j \geq d - 1$ via the evolution and constraint equations. Thus, this tensor carries the full information about the metric g_{ab} which is not already supplied by the boundary conditions, i.e., the “non-kinematical” information. The tensor $(\tilde{p}_a{}^b)_{d-2}$ is directly

¹⁰Note that the the right side of this equation also vanishes, since it vanishes in pure AdS, and since all the coefficients appearing on the right side at this order are identical to the ones in pure AdS. This is a non-trivial consistency check for our asymptotic conditions on the metric.

related to the leading order electric part of the unphysical Weyl tensor, as we will now show.

From the definition of the tensor field \tilde{S}_{ab} , we have

$$\tilde{R}_{abcd} = \tilde{C}_{abcd} + \tilde{g}_{a[c}\tilde{S}_{d]b} - \tilde{g}_{b[c}\tilde{S}_{d]a}. \quad (52)$$

Einstein's equation tells us that $\tilde{S}_{ab} = 2\Omega^{-1}\tilde{K}_{ab}$ for a conformal factor satisfying (37). We substitute this relation into eq. (52), then contract the resulting identity into $\tilde{n}^b\tilde{n}^d$ and project the remaining free indices with $\tilde{h}_a{}^b$, yielding

$$\tilde{h}^e{}_a\tilde{h}^f{}_c\tilde{R}_{ebfd}\tilde{n}^b\tilde{n}^d = \tilde{h}^e{}_a\tilde{h}^f{}_c\tilde{C}_{ebfd}\tilde{n}^b\tilde{n}^d + \Omega^{-1}\tilde{K}_{ac}. \quad (53)$$

Now, from the definition of the Riemann tensor and the extrinsic curvature tensor, we have

$$\begin{aligned} \tilde{h}^e{}_a\tilde{h}^f{}_c\tilde{R}_{ebfd}\tilde{n}^b\tilde{n}^d &= \tilde{h}^e{}_a\tilde{h}^f{}_c\tilde{n}^b(\tilde{\nabla}_e\tilde{\nabla}_b - \tilde{\nabla}_b\tilde{\nabla}_e)\tilde{n}_f \\ &= \mathcal{L}_{\tilde{n}}\tilde{K}_{ac} + \tilde{K}_{ab}\tilde{K}^b{}_c. \end{aligned} \quad (54)$$

Substituting this into eq. (53) yields the equation

$$\tilde{C}^a{}_{bcd}\tilde{n}^b\tilde{n}^d = \tilde{h}^{ea}\tilde{h}^f{}_c\tilde{C}_{ebfd}\tilde{n}^b\tilde{n}^d = -\tilde{K}^a{}_b\tilde{K}^b{}_c + \mathcal{L}_{\tilde{n}}\tilde{K}^a{}_c - \Omega^{-1}\tilde{K}^a{}_c. \quad (55)$$

We can expand this equation in powers of Ω , and remember that the Lie derivative with respect to \tilde{n}^a is identical to ∂_Ω for conformal factors satisfying (37). We thereby obtain equations for the expansion coefficients. At order $d-3$, we find the relation

$$\frac{1}{d-3}(\tilde{C}_{abcd}\tilde{n}^c\tilde{n}^d)_{d-3} = (\tilde{K}_{ab})_{d-2} + \frac{1}{d-3}\sum_{m=0}^{d-3}(\tilde{K}_{ac})_m(\tilde{K}_b{}^c)_{d-3-m}. \quad (56)$$

Since, \tilde{h}_{ab} is given by eq. (38) for our choice of conformal factor in AdS space, we know that $(\tilde{h}_{ab})_{d-1}$ vanishes in that case (assuming $d \geq 6$). Consequently, $(\tilde{K}_{ab})_{d-2}$ also vanishes in that case. Therefore, since the Weyl tensor vanishes in pure AdS, the sum on the right side of eq. (56) has to vanish in pure AdS. However, the coefficients appearing in the sum are the same in any asymptotically AdS spacetime when the conformal frame is chosen such that boundary metric $(h_0)_{ab}$ given by the Einstein static universe. Thus, the sum must in fact vanish in all such cases. Consequently, we obtain the relation

$$\frac{1}{d-3}(\tilde{C}_{abcd}\tilde{n}^c\tilde{n}^d)_{d-3} = (\tilde{K}_{ab})_{d-2} \quad \text{when } d \geq 6 \quad (57)$$

for any asymptotically AdS spacetime in the appropriate conformal frame.

The remaining cases $d = 4, 5$ can be treated as follows: In $d = 4$, the sum on the right side of eq. (56) reduces to $2(\tilde{K}_{ac})_1(\tilde{K}^c_b)_0$, which vanishes by eq. (44). In $d = 5$, the sum reduces to $\frac{1}{2}(\tilde{K}_{ac})_1(\tilde{K}^c_b)_1$, which can be expressed in terms of boundary data by the evolution equation. A direct calculation using eqs. (46, 47, 48) gives $\frac{1}{2}(\tilde{K}_{ac})_1(\tilde{K}^c_b)_1 = \frac{1}{8}(\tilde{h}_{ab})_0$, therefore

$$\frac{1}{2}(\tilde{C}_{abcd}\tilde{n}^c\tilde{n}^d)_2 - \frac{1}{8}(h_{ab})_0 = (\tilde{K}_{ab})_3 \quad \text{for } d = 5. \quad (58)$$

Equations (58, 57) give the desired relation between the electric part of the Weyl tensor at order Ω^{d-3} , and $(\tilde{K}_{ab})_{d-2}$ (and therefore also to $(\tilde{p}_a{}^b)_{d-2}$) for our choice of conformal factor.

Let us now look at recursion relation (48) for $j = d - 1$. This can be written as

$$(\tilde{h}_{ab})_{d-1} = -\frac{2}{d-1}(\tilde{K}_{ab})_{d-2} = -\frac{2}{d-1}(\tilde{E}_{ab})_0 \quad \text{for } d = 4, d \geq 6, \quad (59)$$

where we have used eq. (57) and the definition of \tilde{E}_{ab} , while

$$(\tilde{h}_{ab})_4 = -\frac{1}{2}(\tilde{E}_{ab})_0 + \frac{1}{16}(\tilde{h}_{ab})_0, \quad \text{for } d = 5. \quad (60)$$

Using eq. (51), we find that the unphysical line element can be written in the form

$$\begin{aligned} d\tilde{s}^2 &= d\Omega^2 - \left(1 + \frac{1}{4}\Omega^2\right)^2 dt^2 + \left(1 - \frac{1}{4}\Omega^2\right)^2 d\sigma^2 \\ &\quad - \frac{2}{d-1}\Omega^{d-1}\tilde{E}_{ij} dx^i dx^j + O(\Omega^d) \quad \text{for } d \geq 5, \end{aligned} \quad (61)$$

while

$$\begin{aligned} d\tilde{s}^2 &= d\Omega^2 - \left(1 + \frac{1}{2}\Omega^2\right) dt^2 + \left(1 - \frac{1}{2}\Omega^2\right) d\sigma^2 \\ &\quad - \frac{2}{3}\Omega^3\tilde{E}_{ij} dx^i dx^j + O(\Omega^4) \quad \text{for } d = 4. \end{aligned} \quad (62)$$

In these expressions, x^i are coordinates on \mathcal{S} , say t and $d-2$ angles parametrizing S^{d-2} , and Ω has been chosen so that eq. (37) holds. This is the second key result of our analysis.

Let us finally derive that the leading order electric part of the Weyl tensor is divergence free. This can be seen in different ways, for example by combining eqs. (57, 58) with the constraint equation. Another way is to notice that, by combining Einstein's equation and the contracted Bianchi-identities, we have

$$0 = \tilde{\nabla}^a (\Omega^{3-d} \tilde{C}_{abcd}), \quad (63)$$

for any choice of the conformal factor. We now contract this equation into $\tilde{n}^b \tilde{n}^c$ we use that $\tilde{C}_{abcd} = O(\Omega^{d-3})$, and that

$$\tilde{\nabla}_a \tilde{n}_b = \frac{1}{d} (\tilde{\nabla}^c \tilde{n}_c) \tilde{g}_{ab} \quad \text{on } \mathcal{I}, \quad (64)$$

which follows from Einstein's equation. From this, one immediately arrives at the relation $\tilde{D}^a \tilde{E}_{ab} = 0$ at \mathcal{I} (for any choice of the conformal factor).

Let us summarize what we have proved so far in a lemma.

Lemma 3.1: Let (M, g_{ab}) be an asymptotically AdS-spacetime satisfying Einstein's equation, with conformal completion $(\tilde{M}, \tilde{g}_{ab}, \Omega)$. Then the unphysical (= physical) Weyl tensor $\tilde{C}^a{}_{bcd}$ is of order $O(\Omega^{d-3})$, and the leading order electric part of the Weyl tensor, \tilde{E}_{ab} , given by eq. (28), satisfies $\tilde{D}^a \tilde{E}_{ab} = 0$ on \mathcal{I} , for any choice of the conformal factor. If the conformal factor is chosen as in eq. (37), then the unphysical metric can be expanded as eqs. (62, 61) near \mathcal{I} .

We are now ready to prove the formula for \mathcal{H}_ξ . Consider a metric g_{ab} satisfying our asymptotic AdS condition. We have seen that we may choose a conformal factor Ω such that the unphysical metric $d\tilde{s}^2$ takes the form eqs. (61,62), and so that the physical metric consequently takes the form $ds^2 = \Omega^{-2} d\tilde{s}^2$. We may view eqs. (61,62) as a ‘‘gauge condition’’ on the metric, i.e., as picking a particular representative in the equivalence class of metrics that are diffeomorphic to g_{ab} . In this view, Ω is then a fixed function on \tilde{M} which is part of the background structure specifying the asymptotic conditions. The (on-shell) metric variations respecting this gauge choice (with Ω now regarded as fixed) therefore take the form

$$\delta g_{ab} = \gamma_{ab} + \mathcal{L}_\eta g_{ab} \quad (65)$$

where the first piece γ_{ab} is a metric variation of the form

$$\gamma_{ab} = -\frac{2}{d-1} \Omega^{d-1} \delta \tilde{E}_{ab} + O(\Omega^d), \quad (66)$$

(for a fixed choice of Ω), and where the second piece is an infinitesimal diffeomorphism generated by an arbitrary vector field η^a respecting the gauge choice, i.e., a diffeo satisfying $\mathcal{L}_\eta ds_0^2 = O(\Omega^d)$, where ds_0^2 is the line element of exact AdS-space (6). Thus,

$$\mathcal{L}_\eta g_{ab} = -\frac{2}{d-1} \Omega^{d-1} \mathcal{L}_\eta \tilde{E}_{ab} + O(\Omega^d). \quad (67)$$

Inserting these expressions into the definition of the symplectic form $\omega(g, \delta_1 g, \delta_2 g)$, we see that $\omega \upharpoonright \mathcal{S} = 0$. Hence, the consistency condition (23) is satisfied, and we conclude by the general arguments given in the previous section that \mathcal{H}_ξ must exist.

Let us now determine the actual form of \mathcal{H}_ξ . For this, the variations (67, 66) may be analyzed separately. Let us calculate δQ_ξ and $\xi \cdot \theta$ for the variation $\delta g_{ab} = \gamma_{ab}$. We can bring Q_ξ in the form

$$(Q_\xi)_{a_1 \dots a_{d-2}} = \frac{1}{8\pi G} \Omega^{1-d} \tilde{\epsilon}_{a_1 \dots a_{d-2} bc} \tilde{n}^b \xi^c + \frac{-1}{16\pi G} \Omega^{2-d} \tilde{\epsilon}_{a_1 \dots a_{d-2} bc} \tilde{g}^{be} \tilde{\nabla}_e \xi^c. \quad (68)$$

We now take a variation of this expression and use the relations $\delta \xi^a = 0$,

$$\delta \tilde{\epsilon}_{ab \dots c} = -\frac{1}{d-1} \Omega^{d-1} \delta \tilde{E}_d^d \tilde{\epsilon}_{ab \dots c} + O(\Omega^d) = O(\Omega^d), \quad (69)$$

$$\delta \tilde{n}^a = -\tilde{g}^{ac} \delta \tilde{g}_{cb} \tilde{n}^b = +\frac{2}{d-1} \Omega^{d-1} \delta \tilde{E}^{ab} \tilde{n}_b + O(\Omega^d), \quad (70)$$

$$\delta(\tilde{g}^{be} \tilde{\nabla}_e \xi^c) = -2\Omega^{d-2} \tilde{n}^{[b} \delta \tilde{E}^{c]}_d \xi^d - \Omega^{d-2} \delta \tilde{E}^{(bc)} n_d \xi^d + O(\Omega^{d-1}). \quad (71)$$

This gives

$$(\delta Q_\xi)_{a_1 \dots a_{d-2}} = \frac{1}{8\pi G} \tilde{\epsilon}_{a_1 \dots a_{d-2} bc} \tilde{n}^b \delta \tilde{E}^c_d \xi^d + O(\Omega). \quad (72)$$

Using the relation

$${}^{(d)}\tilde{\epsilon} = \tilde{n} \wedge {}^{(d-1)}\tilde{\epsilon} = \tilde{n} \wedge \tilde{u} \wedge {}^{(d-2)}\tilde{\epsilon} \quad (73)$$

between the d -dimensional volume form, the induced $(d - 1)$ -dimensional volume form on the boundary \mathcal{S} and the $(d - 2)$ -dimensional volume form on C , we can rewrite this as

$$(\delta Q_\xi)_{a_1 \dots a_{d-2}} = \frac{1}{8\pi G} \delta \left[{}^{(d-2)}\tilde{\epsilon}_{a_1 \dots a_{d-2}} \tilde{E}^c{}_d \tilde{u}_c \xi^d \right], \text{ on } \mathcal{S}. \quad (74)$$

A similar calculation with the fact $g^{ab} \delta g_{bc} = O(\Omega^{d-1})$, $\delta g^c{}_c = O(\Omega^d)$ gives $\theta(g, \delta g = \gamma) \upharpoonright \mathcal{S} = 0$. Thus, eq. (27) gives

$$\delta \mathcal{H}_\xi = \int_C \delta Q_\xi = \frac{-1}{8\pi G} \delta \int_C \tilde{E}_{ab} \tilde{u}^b \xi^a d\tilde{S}. \quad (75)$$

which is the defining relation for \mathcal{H}_ξ (if ℓ is restored).

Since we have established that the integral on the right-hand side is conserved, it is unchanged by any diffeomorphism which acts on both the metric g_{ab} and the asymptotic symmetry ξ^a . As a result, the variation of this integral under $\delta g_{ab} = \mathcal{L}_\eta g_{ab}$ (while holding ξ^a fixed) is given by replacing ξ^a with $[\eta, \xi]^a$. On the other hand, it can be verified directly from eq. (67) and the definition of the symplectic structure (17) that $\mathcal{H}_{[\xi, \eta]} = \sigma_\Sigma(g; \mathcal{L}_\xi g, \mathcal{L}_\eta g)$. Thus equation (27) does indeed satisfy the variational condition (19) for such variations. As a result, we have shown that \mathcal{H}_ξ given by eq. (27) is the correct expression for the Hamiltonian generator.

4 Comparison to other definitions of conserved quantities

As explained in the previous subsection, our definition of the conserved quantities \mathcal{H}_ξ associated with asymptotic symmetries ξ^a agrees with the one proposed by Ashtekar et al. However, there also exist other definitions of conserved charges associated with asymptotically AdS spacetimes in the literature. In the following subsections, we investigate the relation between some of those charges and our definition above.

4.1 The counterterm subtraction method

In the counterterm subtraction method [14, 15, 16, 17, 18, 19, 20, 21, 22, 23], “charges” \mathcal{Q}_ξ associated with asymptotic symmetries ξ^a are constructed from

an “effective energy momentum tensor”, τ_{ab} , which is obtained by varying an auxiliary “effective boundary Lagrangian” (not to be confused with the Einstein Lagrangian given in eq. (1)). They are defined by

$$\mathcal{Q}_\xi = \lim_{C \rightarrow \mathcal{I}} \int_C \tau_{ab} \xi^a \hat{u}^b dS, \quad (76)$$

where C is a sequence of cross sections that is taken to \mathcal{I} within a Cauchy surface Σ , and where \hat{u}^a is the unit normal to that surface. The charges \mathcal{Q}_ξ are quite different, both in appearance (see below) and conceptually, from the Hamiltonian charges \mathcal{H}_ξ , so it is natural to investigate the relation between the two. A first step in this direction was taken by Ashtekar and Das [6], who derived an explicit expression for the difference between \mathcal{H}_ξ and \mathcal{Q}_ξ in terms of the extrinsic curvature of \mathcal{I} in the ambient manifold \tilde{M} . An explicit evaluation of this expression for pure AdS space shows that \mathcal{H}_ξ and \mathcal{Q}_ξ differ, as also remarked in [15, 6]. However, the analysis of [6] left open the question whether the difference between \mathcal{Q}_ξ and \mathcal{H}_ξ would in general be depend on the particular asymptotic AdS spacetime under consideration, or whether that difference would only consist of a constant offset. In addition, one may ask what algebra the charges \mathcal{Q}_ξ generate under Poisson bracket. We will now address these issues.

The actual form of the effective boundary Lagrangian, and of τ_{ab} , depends on the number of spacetime dimensions, and becomes increasingly complicated for increasing d . To keep our discussion as simple as possible, and because of the relevance of that case to the AdS/CFT-correspondence, we restrict attention to the case $d = 5$ (as is also done in [6]). However, our arguments could, in principle, be extended to higher dimensions and one would expect a similar result. A simple independent argument addressing the algebra generated by the \mathcal{Q}_ξ in all dimensions will appear in [35].

Let (M, g_{ab}) be an asymptotically AdS spacetime in the sense of our definition, satisfying the Einstein equations. Let $(\tilde{M}, \tilde{g}_{ab}, \Omega)$ be its conformal completion. We denote by \mathcal{I}_Ω the surfaces of constant Ω , so that, when Ω is small, the \mathcal{I}_Ω are timelike surfaces. Let $\hat{\eta}^a$ be the inward directed unit normal to \mathcal{I}_Ω , let h_{ab} the intrinsic metric on \mathcal{I}_Ω defined by

$$h_{ab} \equiv -\hat{\eta}_a \hat{\eta}_b + g_{ab}, \quad (77)$$

let $K_{ab} = -h_a^c h_b^d \nabla_c \hat{\eta}_d$ be the extrinsic curvature, and let \mathcal{R}_{ab} the Ricci tensor of h_{ab} on \mathcal{I}_Ω .

The total action for 5-dimensions is given by the Einstein-Hilbert (1) plus the boundary lagrangian

$$\frac{1}{8\pi G} \sqrt{-h} K d^4x - \frac{1}{8\pi G} \sqrt{-h} \left(3 + \frac{1}{4}\mathcal{R}\right) d^4x, \quad (78)$$

where the first term corresponds to the familiar ‘‘Gibbons-Hawking’’ boundary term that may be added to the action of general relativity, while the second and third term are quantities that are constructed entirely out of the intrinsic geometry of \mathcal{S}_Ω . The variation of the total action with respect to the metric h_{ab} provides the effective stress energy tensor

$$\tau_{ab} = \frac{1}{8\pi G} \left[\frac{1}{2} \left(\mathcal{R}_{ab} - \frac{1}{2} \mathcal{R} h_{ab} \right) - 3h_{ab} - K_{ab} + K h_{ab} \right]. \quad (79)$$

Let us introduce

$$\tilde{f} = \Omega^{-2} (1 - \tilde{\eta}^a \tilde{\nabla}_a \Omega), \quad (80)$$

where $\tilde{\eta}^a$ is the inward unit normal to \mathcal{S}_Ω with respect to the metric \tilde{g}_{ab} , and let us express the Gauss-Codacci equation $C_{abcd} \hat{\eta}^c \hat{\eta}^d = -\mathcal{R}_{ab} + K K_{ab} - K_{ac} K^c_b - (d-2)h_{ab}$ in terms of the corresponding unphysical quantities by using Einstein’s equations. Then, using those expressions, one finds

$$\begin{aligned} \tau_{ab} - \frac{-1}{16\pi G} C_{abcd} \hat{\eta}^c \hat{\eta}^d &= \frac{1}{16\pi G} \left[\tilde{K} \tilde{K}_{ab} - \tilde{K}_a^c \tilde{K}_{cb} + \frac{1}{2} (\tilde{K}_{cd} \tilde{K}^{cd} - \tilde{K}^2) \tilde{h}_{ab} \right. \\ &\quad \left. - 2\Omega \tilde{f} (\tilde{K}_{ab} - \tilde{K} \tilde{h}_{ab}) - 3\Omega^2 \tilde{f}^2 \tilde{h}_{ab} \right] \end{aligned} \quad (81)$$

$$\equiv \Omega^2 \tilde{\Delta}_{ab}, \quad (82)$$

where we remind the reader that tilda quantities refer to the unphysical spacetime, i.e., $\tilde{h}_{ab} = \Omega^2 h_{ab}$ and $\tilde{K}_{ab} = -\tilde{h}^c_a \tilde{h}^d_b \tilde{\nabla}_c \tilde{\eta}_d$. From eqs. (44, 35) we know that $\tilde{K}_{ab} \upharpoonright \mathcal{S} = 0 = \tilde{f} \upharpoonright \mathcal{S}$, implying that $\tilde{\Delta}_{ab}$ is finite at \mathcal{S} . The difference between the counterterm charge and the Hamiltonian charge can immediately be found by integrating eq. (81) over C . It is given by

$$\mathcal{Q}_\xi - \mathcal{H}_\xi = \int_C \tilde{\Delta}_{ab} \tilde{u}^b \xi^a d\tilde{S}, \quad (83)$$

where \tilde{u}^a is the unit normal (normalized with respect to \tilde{g}_{ab}) to C within \mathcal{S} . This is the expression obtained by Ashtekar and Das [6].

To understand better the structure of the integrand $\tilde{\Delta}_{ab}$, one needs to further use Einstein's equations. For this, we note that the unphysical metric can be written as

$$\tilde{g}_{ab} = (1 - \Omega^2 \tilde{f})^{-2} \tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega + \tilde{h}_{ab}, \quad (84)$$

where $\tilde{n}^a = \tilde{\nabla}^a \Omega$, and $\tilde{h}_{ab} \tilde{n}^b = 0$. Using expansion techniques similar to those in the previous section, one can infer from Einstein's equations that

$$\Omega^{-1} \tilde{K}_{ab} = \frac{1}{2} \left(\tilde{\mathcal{R}}_{ab} - \frac{1}{6} \tilde{\mathcal{R}} \tilde{h}_{ab} \right) + \tilde{f} \tilde{h}_{ab} \quad \text{on } \mathcal{I}. \quad (85)$$

Substituting this back into eq. (81), one gets

$$\tilde{\Delta}_{ab} = \frac{1}{64\pi G} \left(\frac{2}{3} \tilde{\mathcal{R}} \tilde{\mathcal{R}}_{ab} - \frac{1}{4} \tilde{\mathcal{R}}^2 \tilde{h}_{ab} - \tilde{\mathcal{R}}_{ac} \tilde{\mathcal{R}}^c_b + \frac{1}{2} \tilde{\mathcal{R}}_{mn} \tilde{\mathcal{R}}^{mn} \tilde{h}_{ab} \right) \quad (86)$$

on \mathcal{I} , where we note that the terms involving \tilde{f} have cancelled. and where $\tilde{\mathcal{G}}_{ab}$ denotes the Einstein tensor on \mathcal{I} . Therefore we find

$$\begin{aligned} \mathcal{Z}_\xi &\equiv \mathcal{Q}_\xi - \mathcal{H}_\xi = \\ &\frac{1}{64\pi G} \int_C \left(\frac{2}{3} \tilde{\mathcal{R}} \tilde{\mathcal{R}}_{ab} - \frac{1}{4} \tilde{\mathcal{R}}^2 \tilde{h}_{ab} - \tilde{\mathcal{R}}_{ac} \tilde{\mathcal{R}}^c_b + \frac{1}{2} \tilde{\mathcal{R}}_{mn} \tilde{\mathcal{R}}^{mn} \tilde{h}_{ab} \right) \xi^a \tilde{u}^b d\tilde{S} \end{aligned} \quad (87)$$

where we note that, so far, we have nowhere assumed that \tilde{h}_{ab} is the metric of the Einstein static universe¹¹. Thus, the charge determined by the counterterm subtraction method disagrees with \mathcal{H}_ξ . However, the difference between the two is just given by a constant offset, which is determined in terms of the boundary metric \tilde{h}_{ab} and its curvature tensor, and which is hence independent of the actual asymptotically AdS solution. It can therefore be evaluated in *any* asymptotically AdS solution with a given boundary metric \tilde{h}_{ab} , in particular in pure AdS space.

For this reason, the difference \mathcal{Z}_ξ has vanishing Poisson bracket with any observable and, in particular, with the generators \mathcal{H}_ξ . Since the generators

¹¹Note that the trace of the integrand of the right side corresponds to the trace of τ_{ab} and is given by $\tau_a^a = \frac{1}{8\pi G} (\frac{1}{8} \tilde{\mathcal{R}}_{ab} \tilde{\mathcal{R}}^{ab} - \frac{1}{24} \tilde{\mathcal{R}}^2)$. This corresponds to the ‘‘trace-anomaly’’ found in [15, 14] in the context of the AdS-CFT correspondence.

\mathcal{H}_ξ satisfy the algebra eq. (20) under Poisson brackets, the charges \mathcal{Q}_ξ satisfy the algebra

$$\{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} = \mathcal{Q}_{[\xi, \eta]} - \mathcal{Z}_{[\xi, \eta]}, \quad (88)$$

i.e., a trivial central extension of the $O(d-1, 2)$ algebra. In addition, we may note that the counter-term energy is consistent with the covariant phase space methods of [24], which controls only variations of the Hamiltonian on the space of solutions.

Let us now use the fact that, on \mathcal{I} , we may take \tilde{h}_{ab} to be the metric of the Einstein static universe, $\tilde{h}_{ab} = -t_a t_b + \sigma_{ab}$ (with $t^a = (\partial/\partial t)^a$). This metric has the Ricci tensor

$$\tilde{\mathcal{R}}_{ab} = 2\sigma_{ab}. \quad (89)$$

Inserting this into eq. (87), we get

$$\mathcal{Q}_\xi - \mathcal{H}_\xi = \frac{1}{16\pi G} \int_C \left(\frac{3}{4} t_a t_b + \frac{1}{4} \sigma_{ab} \right) \xi^a \tilde{u}^b d\tilde{S}. \quad (90)$$

Choosing the symmetry to be a time translation, $\xi^a = t^a$, gives

$$\mathcal{Q}_t - \mathcal{H}_t = \frac{3A_3}{64\pi G} \quad (91)$$

where A_3 is the area of the unit 3-sphere. In particular, while \mathcal{H}_t vanishes in exact AdS space, the charge \mathcal{Q}_t does not vanish and is given by the right side of the above equation. In the context of the AdS/CFT correspondence, the above value of \mathcal{Q}_t in pure AdS space is interpreted as the Casimir energy of the CFT. However, our result implies the stronger statement that, in the above conformal frame, $\mathcal{H}_t - \mathcal{Q}_t$ is given by expression (90) in *any* asymptotically AdS spacetime.

We finally note that \mathcal{Q}_ξ is conserved for any asymptotic symmetry ξ^a , in any asymptotic AdS spacetime, in the sense that it does not depend on the cross section C chosen to calculate \mathcal{Q}_ξ . This follows from the fact that \mathcal{H}_ξ has this property, and the fact that the integrand $\frac{3}{4}t_a t_b + \frac{1}{4}\sigma_{ab}$ on the right side of eq. (87) is covariantly conserved on \mathcal{I} , and has a vanishing trace¹².

¹²Note that there is a claim to the opposite on p.13 in [6]; we suspect a calculation error in this reference.

4.2 The Henneaux-Teitelboim definition

Henneaux and Teitelboim [7] consider asymptotically AdS boundary conditions on the metric specified by demanding that there exists a coordinate system $x^\mu = (t, r, \theta^i)$ near infinity (with θ^i coordinates on S^{d-2}) such that the line element ds^2 under consideration can be written as

$$ds^2 = ds_0^2 + \sum_{\mu, \nu} \gamma_{\mu\nu} dx^\mu dx^\nu \quad (92)$$

where ds_0^2 is the line element of exact AdS space given by eq. (4), and where it is demanded that

$$\gamma_{tt} = O(r^{-d+3}), \quad (93)$$

$$\gamma_{rr} = O(r^{-d-1}), \quad (94)$$

$$\gamma_{tr} = O(r^{-d}), \quad (95)$$

$$\gamma_{r\theta^i} = O(r^{-d}), \quad (96)$$

$$\gamma_{t\theta^i} = O(r^{-d+3}), \quad (97)$$

$$\gamma_{\theta^i\theta^j} = O(r^{-d+3}). \quad (98)$$

If ξ^a is a Killing vector field of exact AdS-space, then the integrals in

$$\begin{aligned} \mathcal{Q}_\xi &= \int_\Sigma \mathcal{C}^a \xi_a \\ &+ \lim_{C \rightarrow \mathcal{S}} \frac{1}{16\pi G} \int_C G_a{}^{bcd} [\xi^e \hat{u}_e D_b \gamma_{cd} - \gamma_{cd} D_b (\xi^e \hat{u}_e)] \hat{\eta}^a dS \\ &+ \lim_{C \rightarrow \mathcal{S}} \frac{1}{4\pi G} \int_C (\kappa_{ab} - \kappa q_{ab}) \xi^a \hat{\eta}^b dS \end{aligned} \quad (99)$$

converge as C tends to a cross section at \mathcal{S} within a spatial slice Σ . Here, \mathcal{C}^a are the constraints (viewed as $(d-1)$ forms), \hat{u}^a is the unit normal to Σ , $\hat{\eta}^a$ is the unit normal¹³ to C within Σ , $q_{ab} = g_{ab} + \hat{u}_a \hat{u}_b$ is the induced metric on Σ , D_a is the associated spatial derivative operator, and $\kappa_{ab} = -q_a{}^c q_b{}^d \nabla_c \hat{u}_d$ is the extrinsic curvature. Moreover,

$$G_{abcd} = \frac{1}{2} (q_{ac} q_{bd} + q_{ad} q_{bc} - 2q_{ab} q_{cd}). \quad (100)$$

¹³In other words, $\hat{u}^{[a} \hat{\eta}^{b]}$ is the binormal to C .

has been defined. Henneaux and Teitelboim take \mathcal{Q}_ξ as the definition of the conserved quantity associated with ξ^a for a spacetime satisfying the asymptotic conditions eqs. (93—98). These conditions are, in turn, motivated by the fact (i) that they hold in the familiar examples of black hole spacetimes in the presence of a cosmological constant, (ii) that they are preserved when acting with a diffeomorphism ψ that is an exact symmetry of the pure AdS background, and (iii) that they imply finiteness of the charges \mathcal{Q}_ξ , and these can be shown to form a representation of the asymptotic symmetry algebra under Poisson brackets.

The asymptotic conditions eqs. (93—98) (including the precise notion of an asymptotic symmetry), and the expression \mathcal{Q}_ξ for the charges are different in appearance from our asymptotic conditions and the charges \mathcal{H}_ξ , but we will now show that they are, in fact equivalent. Starting with the boundary conditions, assume that (M, g_{ab}) satisfies the asymptotic conditions proposed by Henneaux and Teitelboim. Then, defining e.g. $\Omega = 1/r$, we see that the metric also satisfies our boundary conditions. Conversely, assume that our boundary conditions hold and that Einstein's equation is satisfied. The conformal factor Ω may then be chosen so that the unphysical metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$ has the form given in eqs. (62, 61). We choose coordinates x^μ as follows: We define a coordinate r by

$$r = \frac{1}{2}(\Omega^{-1} - \Omega), \quad (101)$$

and on \mathcal{S} . We choose coordinates (t, θ^α) such that, on \mathcal{S} , the line element of the metric \tilde{h}_{ab} takes the form $-dt^2 + \sum \sigma_{ij} d\theta^i d\theta^j$, where σ_{ij} are the coordinate components of the round metric on S^{d-2} . A point x in a neighborhood of \mathcal{S} may then be assigned coordinates $x^\mu = (t, r, \theta^i)$ in an arbitrary smooth manner. Substituting r in terms of Ω in eqs. (62, 61) then immediately implies that $ds^2 = \Omega^{-2} d\tilde{s}^2$ can be written as $ds_0^2 + \sum \gamma_{\mu\nu} dx^\mu dx^\nu$, where the coordinate components $\gamma_{\mu\nu}$ have the fall off given in eqs. (93—98). Thus, our boundary conditions and the ones considered by Henneaux and Teitelboim are equivalent.

Let us compare next the respective notions of asymptotic symmetry. Let g_{ab} be an asymptotically AdS metric, written in the form (92) for some choice of coordinates x^μ . Henneaux and Teitelboim consider vector fields ξ^a that are exact symmetries of the underlying background AdS space chosen, implying that $\mathcal{L}_\xi g_{ab} = \mathcal{L}_\xi \gamma_{ab}$. The components $\mathcal{L}_\xi \gamma_{\mu\nu}$ can be checked to satisfy the fall

off conditions given in eqs. (93—98). Therefore, the 1-parameter group ψ_t of diffeos generated by ξ^a has the property that if g_{ab} is asymptotically AdS, then so is $\psi_t^* g_{ab}$. Consequently, ξ^a is an asymptotic symmetry in our sense. Conversely, let ξ^a be a vector field on M that is an asymptotic symmetry in our sense, let g_{ab} the metric of exact AdS space, and let x^μ be coordinates such that $ds_0^2 = \sum g_{\mu\nu} dx^\mu dx^\nu$ takes the form (4). Now it is easy to see that, in a neighborhood of \mathcal{S} , one can find a diffeomorphism ϕ leaving \mathcal{S} invariant such that ξ^a is a Killing field for $\phi^* g_{ab}$. Now let x'^μ be the coordinates related to x^μ via ϕ . It follows that ξ^a is a Killing field of $\sum (g_{\mu\nu} \circ \phi) dx'^\mu dx'^\nu$, which has the form (4). Consequently, ξ^a is an asymptotic symmetry in the sense of Henneaux and Teitelboim.

Let us finally compare the charges \mathcal{Q}_ξ to our charges \mathcal{H}_ξ . The charges \mathcal{H}_ξ are uniquely determined by the requirement that they satisfy eq. (21) and that $\mathcal{H}_\xi = 0$ in exact AdS space [24]. However, the charges \mathcal{Q}_ξ of Henneaux and Teitelboim are constructed precisely so as to satisfy these conditions as well [7]. Hence they must agree with our charges. Note that, since the definition of \mathcal{H}_ξ is manifestly “background independent”, this result implies that the same is true for the charges \mathcal{Q}_ξ . even though the latter are not manifestly background independent, due to the dependence of γ_{ab} on the choice of coordinates x^μ implicit in eq. (92).

4.3 The spinor definition

Another way to define conserved charges associated with energy and momentum is provided by the spinor method [13, 12, 11]. In this approach, one makes use of the fact that the asymptotic symmetry group has a spinor representation. Consider an asymptotically AdS spacetime (M, g_{ab}) , and assume that a spinor bundle, S , and corresponding curved space gamma matrices γ_a satisfying $\gamma_{(a}\gamma_{b)} = g_{ab}$ can be defined¹⁴. Given an asymptotic symmetry represented by a future directed, timelike or null vector field ξ^a , one can find an auxiliary spinor field ψ such that $\xi^a = \overline{\psi}\gamma^a\psi$, up to terms that vanish at infinity. One then considers the “Nester 2-form”

$$B_{ab} = \frac{1}{8\pi G} \left[\overline{\psi}\gamma_{[a}\gamma_b\gamma_{c]}\widehat{\nabla}^c\psi - \overline{(\widehat{\nabla}^c\psi)}\gamma_{[a}\gamma_b\gamma_{c]}\psi \right], \quad (102)$$

¹⁴Here one needs to assume that (M, g_{ab}) admits a spinor bundle, which imposes some mild restrictions on the topology of M .

where overline denotes the Dirac conjugate of a spinor field, and where $\widehat{\nabla}_a$ is the operator defined by

$$\widehat{\nabla}_a \psi = \nabla_a \psi - \frac{1}{2\ell} \gamma_a \psi \quad (103)$$

in terms of the covariant derivative operator on spinor fields. (From now on we set $\ell = 1$.) The spinor charge is defined by¹⁵

$$\mathcal{Q}_\xi = \lim_{C \rightarrow \mathcal{I}} \int_C B_{ab} \hat{u}^b \hat{\eta}^a dS, \quad (104)$$

where \hat{u}^a is the unit normal to the Cauchy surface Σ in which the $(d-2)$ surfaces C are embedded and where $\hat{\eta}^a$ is the normal to C within Σ . In order for the limit to exist, the spinor field ψ must be chosen in such a way that $\widehat{\nabla}_a \psi$ vanishes sufficiently fast at infinity. Finally, one may prove that ψ can, in addition, be chosen to satisfy the ‘‘Witten condition’’, $q^{ab} \gamma_a \widehat{\nabla}_b \psi = 0$ on Σ (with q_{ab} given by $g_{ab} = -\hat{u}_a \hat{u}_b + q_{ab}$). If this condition is imposed¹⁶ then \mathcal{Q}_ξ can be brought into a form which is manifestly non-negative, and which vanishes if and only if the spacetime under consideration is pure AdS. Therefore, since the future directed, timelike or null asymptotic symmetries ξ^a correspond to energy-momentum, it follows that the charges \mathcal{Q}_ξ give positive energy-momentum for any asymptotically AdS spacetime, and it shows that the only spacetime with zero energy is AdS-space itself.

It was shown in [28] that the charge \mathcal{Q}_ξ agrees with the Ashtekar-Magnon definition, and hence with \mathcal{H}_ξ in $d = 4$ dimensions. This establishes in particular that \mathcal{H}_ξ is a non-negative quantity whenever ξ^a is a future pointing timelike or null asymptotic symmetry, and that $\mathcal{H}_\xi = 0$ implies that the spacetime is (isometric to) pure AdS. We will now show that $\mathcal{H}_\xi = \mathcal{Q}_\xi$ also in higher dimensions, thereby showing that \mathcal{H}_ξ yields an energy that is positive.

Consider first the case when the metric g_{ab} is that of exact AdS space. Then it can be shown that there exist d linearly independent spinor fields α

¹⁵In differential forms notation, $\mathcal{Q}_\xi = \int_C * B$.

¹⁶If Σ has ‘‘inner boundaries’’ corresponding to horizon cross sections of black holes, then suitable boundary conditions need to be imposed upon ψ at those inner boundaries in addition to the conditions upon the asymptotic behavior of ψ at infinity. Those may be taken to be of the form $\epsilon^{ab} \gamma_a \gamma_b \psi \upharpoonright \mathcal{B} = 0$, where ϵ^{ab} is the binormal to the horizon cross section \mathcal{B} , see [11].

satisfying the ‘‘Killing spinor equation’’

$$\widehat{\nabla}_a \alpha = 0 \quad (\text{exact AdS}). \quad (105)$$

The vector field $\xi^a = \bar{\alpha} \gamma^a \alpha$ can then be seen to be a (necessarily timelike or null) Killing vector field in AdS. The spinor charge \mathcal{Q}_ξ by definition vanishes for these symmetries, i.e., the energy-momentum of pure AdS computed with the spinor charge vanishes, in agreement with the charges \mathcal{H}_ξ .

In order to investigate the relation between \mathcal{Q}_ξ and \mathcal{H}_ξ in general asymptotically AdS spacetimes, we pass to a conformal completion $(\tilde{M}, \tilde{g}_{ab}, \Omega)$ with corresponding spinor bundle \tilde{S} and gamma matrices $\tilde{\gamma}_{(a} \tilde{\gamma}_{b)} = \tilde{g}_{ab}$. Spinor quantities (i.e., sections) in \tilde{S} or its tensor products can naturally be identified with sections in S by putting $\tilde{\psi} = \Omega^{1/2} \psi$ and $\tilde{\gamma}_a = \Omega \gamma_a$. With these identifications, it then follows that

$$\overline{\tilde{\psi}} \tilde{\gamma}^a \tilde{\psi} = \overline{\psi} \gamma^a \psi, \quad (106)$$

and it follows that $\tilde{\gamma}_a$ is smooth at \mathcal{I} . The relation between the physical and unphysical derivative operators on spinors is given by

$$\nabla_a \psi = \tilde{\nabla}_a \psi + \frac{1}{2} \Omega^{-1} (\tilde{\gamma}_a \tilde{\gamma}_b \tilde{n}^b + \tilde{n}_a) \psi. \quad (107)$$

As shown above, it is always possible to write the metric in the form (61, 62) when Einstein’s equations are satisfied. The first line in these expressions is the metric $d\tilde{s}_0^2 = \Omega^2 ds_0^2$ of pure AdS space multiplied by a conformal factor. We choose $\psi = \alpha$, where α is a Killing spinor of pure AdS. Then $\tilde{\psi} = \Omega^{1/2} \psi$ is smooth at \mathcal{I} , and using the fact that ψ is a Killing spinor for ds_0^2 , it can then be verified that

$$\widehat{\nabla}_a \psi = -\frac{1}{2} \Omega^{d-5/2} \tilde{\gamma}_c \tilde{\gamma}_d \tilde{n}^{[c} \tilde{E}^d]_a \tilde{\psi} - \frac{1}{2(d-1)} \Omega^{d-5/2} \tilde{\gamma}_c \tilde{E}^c{}_a (\tilde{\gamma}_b \tilde{n}^b - 1) \tilde{\psi} + O(\Omega^{d-3/2}). \quad (108)$$

It also follows from the fact that ψ is a Killing spinor for the pure AdS metric, that

$$\tilde{n}^a \tilde{\gamma}_a \tilde{\psi} \upharpoonright \mathcal{I} = \tilde{\psi} \upharpoonright \mathcal{I}. \quad (109)$$

Substituting this expression into eq. (108), we get

$$\widehat{\nabla}_a \psi = \frac{1}{2} \Omega^{d-5/2} \tilde{\gamma}_c \tilde{E}^c{}_a \tilde{\psi} + O(\Omega^{d-3/2}). \quad (110)$$

We now substitute this into the definition of B_{ab} , we contract the result with $\hat{\eta}^a \hat{u}^b$, we use that $\Omega \tilde{u}^a = \hat{u}^a$ and $\Omega \tilde{n}^a = \hat{\eta}^a$ on \mathcal{I} , and we use the gamma-matrix identity

$$\tilde{\gamma}^{[b} \tilde{\gamma}^a \tilde{\gamma}^{d]} = \tilde{\gamma}^{[b} \tilde{\gamma}^a] \tilde{\gamma}^d + 2\tilde{g}^{d[b} \tilde{\gamma}^{a]}. \quad (111)$$

If this is done, one finds

$$B_{ab} \hat{u}^a \hat{\eta}^b = -\frac{1}{8\pi G} \Omega^{d-2} (\overline{\tilde{\psi}} \tilde{\gamma}^c \tilde{\psi}) \tilde{E}_{cd} \tilde{u}^d + O(\Omega^{d-1}). \quad (112)$$

We finally integrate this equation over C and use that $dS = \Omega^{2-d} d\tilde{S}$, as well as $\xi^a = \overline{\tilde{\psi}} \tilde{\gamma}^a \tilde{\psi}$. This gives

$$\mathcal{Q}_\xi = \lim_{C \rightarrow \mathcal{I}} \int_C B_{ab} \hat{u}^a \hat{\eta}^b dS = \frac{-1}{8\pi G} \int_C \tilde{E}_{ab} \tilde{u}^b \xi^a d\tilde{S} = \mathcal{H}_\xi, \quad (113)$$

proving the equivalence of \mathcal{Q}_ξ and \mathcal{H}_ξ in d dimensions.

5 Inclusion of matter fields

We now apply the formalism developed in sections 2 and 3 to derive the explicit formula for \mathcal{H}_ξ in the case where additional (bosonic) matter fields are present. The matter fields of most interest are those which appear in the supergravity theories relevant to various AdS/CFT conjectures; i.e., to scalars and anti-symmetric tensor fields. An overview of the results is presented in section 5.1, followed by a detailed treatment of the conserved energy in the presence of such fields using the covariant phase space approach (section 5.2), the counter-term subtraction approach (section 5.3), and the spinor approach (section 5.4).

5.1 Overview of results with matter fields

The effect of matter fields on \mathcal{H}_ξ is determined by the contributions of such fields to the Noether charge Q_ξ and to the symplectic potential θ . Let us first discuss the anti-symmetric tensor fields. We restrict to such fields $A_{a_1 \dots a_p}$ with non-negative mass-squared terms (defined in Appendix B) and with the standard Maxwell-type kinetic term, noting that all anti-symmetric tensors in the S^5 compactification of ten-dimensional type IIB supergravity [39, 40]

to AdS_5 and in the S^4 compactification of eleven-dimensional supergravity [41] to AdS_7 to satisfy this criterion¹⁷. One finds a contribution to Q_ξ proportional to $*F(\xi \cdot A)$ (where $F = dA$) and a contribution to θ proportional to $*F \wedge \delta A$.

Whether or not these contributions are large enough to affect \mathcal{H}_ξ depends on the asymptotic conditions satisfied by the fields. As for the case of gravitational fluctuations, we defer a detailed treatment of the linearized equations of anti-symmetric tensor fields to Appendix B. Under the conditions stated above, Appendix B shows that for a field of given mass and given rank there is a unique boundary condition such that the evolution operator is self-adjoint in a natural inner product. For such boundary conditions, the fall-off at infinity of anti-symmetric tensor fields is too fast to contribute to \mathcal{H}_ξ or to affect the analysis of sections 2 and 3.

Let us therefore examine the case of scalar fields, where the situation is somewhat more subtle. Consider a scalar field with Lagrange density

$$L^{\text{matter}} = -\frac{1}{2}d^d x \sqrt{-g}[\nabla^a \phi \nabla_a \phi + V(\phi)], \quad (114)$$

where $V(\phi)$ is a potential¹⁸. To quadratic order in ϕ , it is given by

$$V(\phi) = m^2 \phi^2 + \dots \quad (115)$$

The prefactor in front of L^{matter} is chosen so that Einstein's equations for the combined gravity and matter Lagrangian $L = L^{\text{grav}} + L^{\text{matter}}$ takes the standard form $R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$, where the matter stress energy tensor is given by

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2}g_{ab}[\nabla^c \phi \nabla_c \phi + V(\phi)]. \quad (116)$$

¹⁷At least, they satisfy this criterion to quadratic order in fields. Under the asymptotic behavior specified in appendix B, cubic and higher terms vanish too quickly at infinity to contribute. Such cubic terms arise from Chern-Simons terms as well as from couplings to scalar fields.

¹⁸We assume below that no curvature couplings are present. The standard presentation of 10- and 11-dimensional supergravity compactified to AdS_5 and AdS_7 is free of such couplings [39, 40, 41]. In general, one may choose to work in the Einstein conformal frame and thereby eliminate any couplings to the Ricci scalar. See, however, [42] for a treatment of the counter-term subtraction approach in a conformal frame with curvature couplings.

Either requiring the energy to be bounded below [32] or the dynamics to be self-adjoint [43] restricts m^2 to satisfy the so-called Breitenlohner-Friedman bound (BF-bound),

$$m^2 \geq -\frac{(d-1)^2}{4\ell^2}. \quad (117)$$

Scalars arising in 10- and 11-dimensional supergravity compactified to AdS_5 and AdS_7 have been explicitly shown [39, 40, 41] to satisfy this bound, though in the AdS_5 case scalars do arise that saturate the bound.

For $0 \geq m^2 > -(d-1)^2/4\ell^2$, solutions to the linearized equations of motion have the characteristic fall-off at infinity

$$\phi \sim \Omega^{\lambda_-} \phi_- + \Omega^{\lambda_+} \phi_+, \quad (118)$$

where ϕ_-, ϕ_+ are functions on \mathcal{S} , and where

$$\lambda_{\pm} = \frac{d-1}{2} \pm \frac{1}{2} \sqrt{(d-1)^2 + 4\ell^2 m^2}. \quad (119)$$

When the BF-bound is saturated, the roots $\lambda_{\pm} = \lambda$ are degenerate and the second solution falls off as $\Omega^{\lambda} \log \Omega$. For masses $m^2 \geq -(d-1)^2/4\ell^2 + 1$, boundedness of the energy [32] or self-adjointness of the evolution [43] requires the faster fall-off asymptotic behavior given by λ_+ , but in the range $-(d-1)^2/4\ell^2 \leq m^2 \leq -(d-1)^2/4\ell^2 + 1$ these requirements impose no restriction. Thus, for such masses any boundary condition compatible with (118) may be imposed.

However, even in the range $-(d-1)^2/4\ell^2 \leq m^2 \leq -(d-1)^2/4\ell^2 + 1$ it turns out that only the most rapidly decreasing asymptotic behavior (associated with λ_+ above the BF-bound, and without logs when the bound is saturated) is compatible with the asymptotic conditions we imposed on the metric in section 2. As a result, we restrict our attention to this rapidly decreasing setting below. The reader may consult [44, 45, 46, 47, 48, 49, 50, 51] for recent work addressing the construction of a conserved energy in settings with slower fall-off at infinity.

For simplicity, we focus in section (5.2) on scalars which saturate the BF-bound. We will see that it is marginally compatible with our analysis of the pure gravitational case. As a result, scalars with faster fall-off decrease too rapidly at infinity to affect the expression for \mathcal{H}_{ξ} , while slower fall-off would

require an extended treatment beyond the scope of the present work. Rather surprisingly, we will find that in the marginal BF-bound saturating case the expression (27) for \mathcal{H}_ξ is unmodified. We will also show in section 5.3 that even in the presence of such fields the counter-term subtraction definition of energy in $d = 5$ continues to differ from \mathcal{H}_ξ by exactly expression (90), just as in the pure gravity case¹⁹.

5.2 Scalar fields saturating the BF-bound and the co-variant phase space construction

Let us now consider in detail scalar fields saturating the BF-bound. Under our rapid fall-off assumption, these scalars satisfy the following **asymptotic condition**: In the unphysical spacetime, there is a function ϕ_0 which is smooth at \mathcal{I} such that

$$\phi = \Omega^{\frac{d-1}{2}} \phi_0. \quad (120)$$

An immediate consequence of this asymptotic condition is that the stress energy tensor is of the same order as the electric part of the Weyl tensor at \mathcal{I} . Therefore, since the latter enters in the expression for \mathcal{H}_ξ in the pure gravity case (see eq. (27)), one would now expect there to be an additional contribution to \mathcal{H}_ξ involving explicitly the scalar field. Surprisingly, a detailed analysis using the same algorithm as above in the pure gravity case shows that such additional contributions are absent, i.e., \mathcal{H}_ξ is given by the same formula in terms of the metric as in the pure gravity case.²⁰ Also remarkably, the Hamiltonian charge \mathcal{H}_ξ is still conserved in the presence of scalars, i.e., it does not depend on the cross section C on which the integral (27) is evaluated. These results rely on a somewhat subtle cancellation between various terms, and we therefore now describe the derivation in some detail.

According to the algorithm specified in section 2, we are instructed to compute the quantities $\xi \cdot \theta(g, \delta g)$, and δQ_ξ for an asymptotic symmetry ξ^a , and then seek \mathcal{H}_ξ as a solution to $\delta \mathcal{H}_\xi = \int_C [\delta Q_\xi - \xi \cdot \theta]$, where C is a cut

¹⁹For a particular class of black hole solutions [52] in which scalars of the above form are excited, and for the conformal frame in which $(h_0)_{ab}$ is the metric of the Einstein static universe, it was shown in [47] that the difference between the counter-term subtraction and Ashtekar et al definitions agrees with this difference evaluated on pure AdS space. The calculations of [48, 53, 54] also suggest such an agreement.

²⁰It would be interesting to investigate whether this continues to be the case under even weaker asymptotic conditions than those explored here.

at infinity, where Q_ξ is the Noether charge of the theory, and where θ is the integrand of the surface term that arises when performing a variation of $L = L^{\text{grav}} + L^{\text{matter}}$ under an integral sign. In the presence of matter fields, each of these quantities generically consists of a gravitational part Q_ξ^{grav} resp. $\xi \cdot \theta^{\text{grav}}$ given by the same formula as in the pure gravity case (see eqs. (11)), as well as contributions Q_ξ^{matter} resp. $\xi \cdot \theta^{\text{matter}}$ from the matter fields. One has

$$\theta_{a_1 \dots a_{d-1}} = \epsilon_{ba_1 \dots a_{d-1}} \left[\frac{1}{16\pi G} (\nabla^c \delta g_c^b - \nabla^b \delta g_c^c) - \delta \phi \nabla^b \phi \right] \quad (121)$$

where the second term is the new contribution from the matter field. However, for minimally coupled scalars²¹ one may show that the Noether charge is given by the same expression as in the pure gravity case, i.e., Q_ξ is given by eq. (22), with no explicit contributions from the matter field.

In order to find \mathcal{H}_ξ we must now analyze in more detail the fall off behavior of the metric and the scalar field ϕ near \mathcal{S} using Einstein's equation. In terms of the unphysical metric \tilde{g}_{ab} , this now takes the form

$$\tilde{S}_{ab} = -2\Omega^{-1} \tilde{\nabla}_a \tilde{n}_b + \Omega^{-2} \tilde{g}_{ab} (\tilde{n}^c \tilde{n}_c - \ell^{-2}) + \tilde{L}_{ab}, \quad (122)$$

where \tilde{S}_{ab} is given by eq. (32), and where \tilde{L}_{ab} is the matter contribution given by

$$\tilde{L}_{ab} \equiv \frac{16\pi G}{d-2} \left[T_{ab} - \frac{1}{d-1} g_{ab} T \right]. \quad (123)$$

To analyze the consequences of Einstein's equation, we choose, as above, the conformal factor so that the unphysical metric takes Gaussian normal form, $\tilde{g}_{ab} = \tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega + \tilde{h}_{ab}$, and perform a power series expansion of eq. (122), putting $\ell = 1$ to simplify the notation. Einstein's equations give recursion relations for the expansion coefficients $(\tilde{h}_{ab})_j$ of the induced metric \tilde{h}_{ab} , the coefficients of the extrinsic curvature $(\tilde{K}_{ab})_j$, and the expansion coefficients of the scalar field, $\phi = \Omega^{(d-1)/2} (\phi_0 + \Omega \phi_1 + \dots)$, which can be solved in a manner similar to the pure gravity case in Sec. 3. By an argument completely analogous to that given in the pure gravity case, one finds that the coefficients

²¹Adding a curvature coupling $L^{\text{curv}} = -\alpha d^d x \sqrt{-g} R \phi^2$ to the matter Lagrangian (114) would, in contrast, generate a correction $-2\alpha (\xi^b \phi \nabla^c \phi) \epsilon_{bca_1 \dots a_{d-2}}$ to the Noether charge (22). Again, this term is finite for scalars satisfying (120). It vanishes for more rapidly decreasing fields and in general diverges for those falling off more slowly.

$(\tilde{h}_{ab})_j$ for $j < d - 1$ and $(\tilde{K}_a{}^b)_l$ for $l < d - 2$ are the same as for the exact AdS metric, while the coefficients $(\tilde{h}_{ab})_{d-1}$ and $(\tilde{K}_a{}^b)_{d-2}$ are related to the leading order electric Weyl tensor and the leading order coefficient of the scalar field²². On the other hand, the higher coefficients ϕ_1, ϕ_2, \dots and $(\tilde{h}_{ab})_j$ for $j > d - 1$ and $(\tilde{K}_a{}^b)_l$ for $l > d - 2$ are uniquely determined by the recursion relations and the leading order electric Weyl tensor, as well as the leading scalar coefficient ϕ_0 .

In order to derive the relation between $(\tilde{h}_{ab})_{d-1}, \phi_0$ and the leading order electric Weyl tensor, one proceeds in the same manner as in the pure gravity case. The arguments leading up to eq. (55) now result in

$$\tilde{C}_{abcd}\tilde{n}^b\tilde{n}^d = \mathcal{L}_{\tilde{n}}\tilde{K}_{ac} - \Omega^{-1}\tilde{K}_{ac} + \tilde{K}_{ab}\tilde{K}{}^b{}_c - \frac{1}{2}\tilde{L}_{bd}\tilde{h}_a{}^b\tilde{h}_c{}^d - \frac{1}{2}\tilde{h}_{ac}\tilde{L}_{bd}\tilde{n}^b\tilde{n}^d \quad (124)$$

which differs only by an additional matter term involving the matter stress tensor \tilde{L}_{ab} . From the asymptotic condition on ϕ , this has an expansion of the form

$$\tilde{L}_{ab} = \Omega^{d-3}(\tilde{L}_{ab})_{d-3} + \Omega^{d-2}(\tilde{L}_{ab})_{d-2} + O(\Omega^{d-1}), \quad (125)$$

where

$$(\tilde{L}_{ab})_{d-3} = 4\pi(d-1)G \left[\phi_0^2\tilde{n}_a\tilde{n}_b - \frac{1}{d-2}\phi_0^2\tilde{h}_{ab} \right] \quad (126)$$

$$(\tilde{L}_{ab})_{d-2} = \frac{8\pi G}{d-2} \left[2(d-1)\phi_0\tilde{n}_{(a}\tilde{\nabla}_{b)}\phi_0 - d \cdot \phi_0\phi_1\tilde{g}_{ab} \right]. \quad (127)$$

We substitute these expressions into eq. (124), and we expand the resulting equation in powers of Ω , as we did above in the pure gravity case. At each order in Ω , this then gives a relation between the expansion coefficients of the electric Weyl tensor, the coefficients $(\tilde{h}_{ab})_j$ of the induced metric, the coefficients of the extrinsic curvature $(\tilde{K}_{ab})_j$, and the expansion coefficients ϕ_j of the scalar field. At order $j = d - 1$, and in dimensions $d = 4, d \geq 6$, the relation can be brought into the form

$$(\tilde{h}_{ab})_{d-1} = -\frac{2}{d-1}(\tilde{E}_{ab})_0 - \frac{4\pi G}{d-2}(\tilde{h}_{ab})_0\phi_0^2, \quad \text{for } d = 4, d \geq 6, \quad (128)$$

²²It is important to note here that the recursion relation eq. (46) at order $j = d - 2$ provides a non-trivial consistency check for the asymptotic condition on the scalar field, because it says that the right side (supplemented with a term proportional to the trace-free part of $\tilde{h}_c{}^a\tilde{h}_b{}^d\tilde{L}_a{}^b$) must vanish. This turns out to be the case, since $\tilde{h}_c{}^a\tilde{h}_b{}^d\tilde{L}_a{}^b$ at this order is a pure trace (see eq. (116)), and because the gravitational contributions cancel as well.

where \tilde{E}_{ab} is the leading order electric Weyl tensor given by eq. (28). In $d = 5$, the corresponding relation turns out to be

$$(\tilde{h}_{ab})_4 = -\frac{1}{2}(\tilde{E}_{ab})_0 - \left(\frac{1}{16} + \frac{4\pi G \phi_0^2}{3}\right) (\tilde{h}_{ab})_0 \quad \text{for } d = 5. \quad (129)$$

One immediately finds from this that unphysical line element must take the form²³

$$\begin{aligned} d\tilde{s}^2 = & d\Omega^2 - \left[\left(1 + \frac{1}{4}\Omega^2\right)^2 - \frac{4\pi G \phi_0^2}{d-2}\Omega^{d-1} \right] dt^2 + \left[\left(1 - \frac{1}{4}\Omega^2\right)^2 - \frac{4\pi G \phi_0^2}{d-2}\Omega^{d-1} \right] d\sigma^2 \\ & - \Omega^{d-1} \frac{2}{d-1} \tilde{E}_{ij} dx^i dx^j + O(\Omega^d) \quad \text{for } d \geq 5. \end{aligned} \quad (130)$$

In $d = 4$, the corresponding result is

$$\begin{aligned} d\tilde{s}^2 = & d\Omega^2 - \left(1 + \frac{1}{2}\Omega^2 - 2\pi G \phi_0^2 \Omega^3\right) dt^2 + \left(1 - \frac{1}{2}\Omega^2 - 2\pi G \phi_0^2 \Omega^3\right) d\sigma^2 \\ & - \frac{2}{3}\Omega^3 \tilde{E}_{ij} dx^i dx^j + O(\Omega^4) \quad \text{for } d = 4. \end{aligned} \quad (131)$$

The rest of the analysis is now similar to the pure gravity case. From the asymptotic form of the metric given above, one finds that the variation of the Noether charge δQ_ξ has exactly the same form as in the pure gravity case (72), even though the metric feels the backreaction of the scalar field, as seen in eqs. (130, 131). The contributions from the scalar field in these expressions happen to cancel out in δQ_ξ . One furthermore calculates that

$$\delta\phi \nabla^b \phi = \Omega^d \left(\frac{d-1}{2} \tilde{n}^b \phi_0 \delta\phi_0 + O(\Omega) \right), \quad (132)$$

and that

$$\frac{1}{16\pi G} (\nabla^c \delta g_c^b - \nabla^b \delta g_c^c) = \Omega^d \left(\frac{d-1}{2} \tilde{n}^b \phi_0 \delta\phi_0 + O(\Omega) \right) \quad (133)$$

It follows from these expressions that the terms involving the contributions involving $\delta\phi_0$ from the matter field precisely cancel out also in the expression

²³Here for simplicity we have as usual assumed that $(\tilde{h}_{ab})_0$ is the standard metric 4 on the Einstein Static Universe so that the conformal factor satisfies (37).

$\xi \cdot \theta$. Consequently, $\delta\mathcal{H}_\xi$ receives no explicit contributions from the matter field and is therefore given by exactly the same expression as in the pure gravity case. Thus

$$\mathcal{H}_\xi = \frac{-1}{8\pi G} \int_C \tilde{E}_{ab} \tilde{u}^b \xi^a d\tilde{S}. \quad (134)$$

Let us finally show that the hamiltonian charges \mathcal{H}_ξ continue to be conserved in the presence of a scalar field saturating the BF-bound. As in the pure gravity case, we may use Gauss' law to express the difference between the generators defined on two different cuts C_1, C_2 as the integral over the enclosed portion \mathcal{S}_{12} of scri of the divergence of the integrand $\tilde{E}_{ab}\xi^b$, see eq. (30). In the pure gravity case, the divergence was shown to be zero. We now show that this is still the case in the presence of the scalar field ϕ . This again follows from Einstein's equation, which in the presence of matter fields leads to the relation

$$\tilde{\nabla}^d \left(\frac{\Omega^{3-d}}{d-3} \tilde{C}_{abcd} \right) = -\Omega^{2-d} [\tilde{L}_{d[a}\tilde{g}_{b]c}\tilde{n}^d + \tilde{\nabla}_{[a}(\Omega\tilde{L}_{b]c})] \quad (135)$$

for any conformal factor Ω . Now contract both sides of the equation into $\tilde{n}^b\tilde{n}^c\tilde{h}^a_e$, and take the limit as $\Omega \rightarrow 0$. On the right side, we get one potentially diverging term coming from $(\tilde{L}_{ab})_{d-3}$ and one converging term coming from $(\tilde{L}_{ab})_{d-2}$ and from $\tilde{\nabla}_{[a}(\tilde{L}_{b]c})_{d-3}$. However, the diverging term is seen to vanish using eq. (126) and taking into account the contractions, while the contributions to the converging term all happen to cancel each other using eqs. (126, 127). Thus, the limit of the contraction of the right side as $\Omega \rightarrow 0$ vanishes. The left side gives $\tilde{D}^a\tilde{E}_{ea}$. Thus, $\tilde{D}^a\tilde{E}_{ea} = 0$ on \mathcal{S} , showing that the charges \mathcal{H}_ξ are conserved in the presence of the matter field ϕ .

5.3 Scalars saturating the BF-bound and the counter-term subtraction notion of energy

From the results above, it is clear that the covariant phase space definition of energy continues to agree with the Ashtekar et al definition of energy in the presence of scalars satisfying the asymptotic condition (118). Furthermore, since our asymptotic conditions on the metric (from section 2) continue to hold, the arguments of section 4.2 once again show that the covariant phase space charge also agrees with the Henneaux-Teitelboim definition.

Let us now compare this definition with the counter-term subtraction method. For scalars saturating the BF-bound (and integer steps above this bound), the counter-term subtraction scheme was analyzed in [20, 21, 22, 23], but we repeat the analysis of the BF-saturating case here for completeness and for consistency with our current notation.

For scalars satisfying (118), the scalar field counter-term required to make the scalar action finite and to yield a well-defined variational principle is given by e.g. [55, 56, 57, 58],

$$-\frac{d-1}{4}d^{d-1}x\sqrt{-h}\phi^2. \quad (136)$$

Thus, the combined effective boundary stress-energy tensor is the sum of the gravitational boundary stress tensor and the stress tensor associated with the scalar boundary Lagrangian. In $d = 5$, this is

$$\tau_{ab} = \frac{1}{8\pi G} \left[\frac{1}{2} \left(\mathcal{R}_{ab} - \frac{1}{2} \mathcal{R} h_{ab} \right) - 3h_{ab} - K_{ab} + K h_{ab} \right] - h_{ab} \phi^2 \quad (137)$$

and the counterterm subtraction charge \mathcal{Q}_ξ in the full theory is again given by eq. (76). Let us compare \mathcal{Q}_ξ with the charge \mathcal{H}_ξ , obtained in the covariant phase space approach. As shown in the previous subsection, the charge \mathcal{H}_ξ is given by exactly the same expression as in the pure gravity case, see eq. (27), and does not contain any explicit contributions from the scalar field. On the other hand τ_{ab} , and hence \mathcal{Q}_ξ , does contain an explicit contribution from the scalar field. Thus, one would naively expect that the difference between \mathcal{Q}_ξ and \mathcal{H}_ξ would depend on the boundary value of the scalar field under consideration. However, as we will now show, this is not the case, and the difference $\mathcal{Q}_\xi - \mathcal{H}_\xi$ is given by exactly the same expression as in the pure gravity case, see eq. (87). This difference is therefore just a constant offset which does not depend on the particular field configuration under consideration, but only on ξ^a and “kinematical” data fixed by the asymptotic conditions.

Let \mathcal{S}_Ω be the timelike surfaces of constant Ω , let h_{ab} , \mathcal{R}_{abcd} and $K_{ab} = -h_a^c h_b^d \nabla_c \hat{\eta}_d$ be the induced metric, the intrinsic Riemann tensor, and the extrinsic curvature, and let $\hat{\eta}^a$ be the unit normal to \mathcal{S}_Ω . Then, from the Gauss-Codacci equations and Einstein’s equation, we have

$$\mathcal{R}_{ac} = L_{bd} h^b_a h^d_c + \frac{1}{2} L h_{ac} - \frac{1}{2} h_{ac} L_{bd} \hat{\eta}^b \hat{\eta}^d - C_{abcd} \hat{\eta}^b \hat{\eta}^d + K K_{ac} - K_{ab} K^b_c - 3h_{ab}, \quad (138)$$

where $L_{ab} = \tilde{L}_{ab}$ is the matter contribution given by eq. (123). Now express the quantities on the right side of this equation in terms of the corresponding unphysical quantities which are given by $h_{ab} = \Omega^{-2}\tilde{h}_{ab}$, $K_{ab} = \Omega^{-1}\tilde{K}_{ab} + \Omega^{-2}\tilde{h}_{ab}$, $\tilde{n}^a = \tilde{\nabla}^a\Omega$, and substitute the resulting expression for \mathcal{R}_{ac} into the definition of τ_{ab} . Then one obtains

$$\begin{aligned} \tau_{ab} - \frac{-1}{16\pi G}C_{abcd}\hat{\eta}^c\hat{\eta}^d &= \frac{-1}{16\pi G}\left[-\tilde{K}\tilde{K}_{ab} + \tilde{K}_a{}^c\tilde{K}_{cb} - \frac{1}{2}(-\tilde{K}^2 + \tilde{K}_{cd}\tilde{K}^{cd})\tilde{h}_{ab}\right. \\ &\quad \left.+ \tilde{L}_{cd}\tilde{h}^c{}_a\tilde{h}^d{}_b - \tilde{L}\tilde{h}_{ab} + \tilde{h}_{ab}\tilde{L}_{cd}\tilde{n}^c\tilde{n}^d\right] - \Omega^2\phi_0^2\tilde{h}_{ab}, \quad (139) \end{aligned}$$

where we are assuming for simplicity that Ω has been chosen so that $\tilde{n}^a\tilde{n}_a = 1$. Now, using the expansion (126) of \tilde{L}_{ab} , we see that the matter terms in the last line precisely cancel (up to order Ω^3). Contracting the above equation into $\xi^a\tilde{u}^b$ (with \tilde{u}^a the unit timelike normal to a cut C of \mathcal{S}), dividing by Ω^2 , and integrating over C , we therefore find that $\mathcal{H}_\xi - \mathcal{Q}_\xi$ is given by exactly the same expression as in the pure gravity case, eq. (83). As in the pure gravity case, this can be brought into the final form (87) using eq. (85) (which does not receive any additional contributions from the scalar field at the relevant order).

5.4 ... and the spinor charge

Let us finally compare the covariant phase space charges with the spinor charge in the presence of a scalar field ϕ with a potential V , saturating the BF-bound. In the pure gravity case, the relevance of the spinor charge is that it has a positivity property. As shown in [59, 60], this positivity property continues to hold in the presence of a single scalar field ϕ if and only if the potential V arises from a superpotential W . More precisely, let

$$P(\phi) = -\frac{1}{8\pi G}(d-1)(d-2) + V(\phi) \quad (140)$$

be the combined scalar field potential and contribution from the cosmological constant (with $\ell = 1$). Then there must be a W such that

$$8\pi G P = -4(d-2)(d-1)W^2 + 4(d-2)^2(\partial W/\partial\phi)^2. \quad (141)$$

The spinor charge is then defined by the same formula as in the pure gravity case [see eq. (104)], but with the $\widehat{\nabla}_a$ -operator now defined by

$$\widehat{\nabla}_a \psi = \nabla_a \psi - W(\phi) \gamma_a \psi. \quad (142)$$

The condition that P be given in terms of W by eq. (141), and that $V = -\frac{1}{4}(d-1)^2 \phi^2 + \dots$ for a field saturating the BF-bound leads to

$$W = \frac{1}{2} + \frac{\pi G(d-1)}{(d-2)} \phi^2 + \dots \quad (143)$$

Let ψ be a Killing spinor in exact AdS space (with $\phi = 0$), and let $\xi^a = \overline{\psi} \gamma^a \psi$. For an asymptotically AdS metric g_{ab} and a scalar field ϕ with the asymptotic behavior $\phi = \Omega^{(d-1)/2} \phi_0$ satisfying Einstein's equation, the asymptotic expansion (130) of the metric holds. From this, and from eq. (109), one obtains

$$\nabla_a \psi - \frac{1}{2} \gamma_a \psi = \Omega^{d-5/2} \tilde{\gamma}^b \tilde{\gamma}^c \left[\frac{1}{2} \tilde{E}_{a[b \tilde{n}_{c]}} + \frac{\pi G(d-1)}{d-2} \phi_0^2 \tilde{h}_{a[b \tilde{n}_{c]}} \right] \tilde{\psi} + O(\Omega^{d-3/2}). \quad (144)$$

Now let \hat{u}^a be the unit future timelike normal to a cross section C in \mathcal{I} . Substituting the above expression into the definition of the Nester 2-form B_{ab} and using eqs. (143, 142) gives

$$\Omega^{2-d} B_{ab} \hat{\eta}^a \hat{u}^b = \frac{-1}{8\pi G} \tilde{E}_{ab} \tilde{u}^a \xi^b + O(\Omega), \quad (145)$$

where $\tilde{u}^a = \Omega \hat{u}^a$. Thus, the explicit contribution from the scalar field entering the spinor charge via eq. (142) is precisely canceled by the indirect contribution of the scalar field to the metric via Einstein's equation. Integrating over C gives that the spinor charge \mathcal{Q}_ξ again agrees with the Hamiltonian charge \mathcal{H}_ξ given by eq. (134) in the presence of scalar fields. From the positivity of the spinor charge we therefore find that \mathcal{H}_ξ also satisfies the positive energy theorem, provided the the scalar field potential arises from a superpotential.

6 Perturbation analysis of the asymptotic behavior of the gravitational field

In the previous sections, we have worked out the consequences of our asymptotic conditions on the metric. The AdS-group is the asymptotic symmetry

group of these conditions, and we showed that gravitational charges associated with asymptotic symmetries can be defined in a clear-cut and natural way, and that the resulting expression agrees with expressions proposed previously in the literature. While these results lend support²⁴ to our choice of asymptotic conditions, they are not, of course, a proof that a “generic” metric will satisfy these conditions, i.e., that there is a sufficiently wide class of solutions satisfying these conditions, nor are they a proof that our asymptotic conditions are the only possible ones.

Unfortunately, due to the nonlinearities of Einstein’s equations, it is difficult to analyze this issue in any straightforward manner. Nevertheless, it is possible to address it in the context of perturbation theory, and we shall do so in this section. At the level of perturbation theory, the corresponding question is as follows: Let γ_{ab} be a solution to the linearized Einstein equation about exact AdS space. Can one extend the perturbed unphysical metric to \mathcal{I} so as to be smooth (or suitably many times differentiable) there modulo a gauge transformation $\mathcal{L}_\eta g_{ab}$, and a change $\delta\Omega$ of the conformal factor Ω of exact AdS space? Thus, given γ_{ab} , we ask whether there is a vector field η^a on pure AdS space, and a function $\delta\Omega$ (with $\delta\Omega = 0$ on \mathcal{I}), such that

$$\tilde{\gamma}_{ab} = \Omega^2 \psi^* (\gamma_{ab} + \nabla_a \eta_b + \nabla_b \eta_a) + \Omega \delta\Omega \psi^* g_{ab} \quad (146)$$

is smooth and vanishing at \mathcal{I} , where ψ is some diffeomorphism. (Here and in the remainder of this section, g_{ab} is the metric of exact AdS-space, and ∇_a is the derivative operator in exact AdS space.) Actually, rather than analyze the perturbed metric, it is much more convenient to analyze instead the perturbed Weyl tensor, δC_{abcd} . This has the triple advantage that the perturbed Weyl tensor is both gauge invariant and conformally invariant, and that it is the key quantity of interest in our formula for the (perturbed) charges of the spacetime. If the Weyl tensor with one upper index $\delta C_{abc}{}^d$ is sufficiently smooth on the unphysical spacetime, then by our previous analysis of the Einstein equations the unphysical metric will be sufficiently smooth. Consequently, by working with δC_{abcd} , we are immediately able to determine whether a generic metric perturbation gives rise to a space time for which the charges \mathcal{H}_ξ can still be defined (to first order).

²⁴Note that a consistency check of the key assumption that the unphysical metric be smooth (i.e., C^∞) was implicit in writing it as a formal power series in Ω in section 3. The key point is that the recursion relations fixing the coefficients of this power series can be solved consistently without the need to, say, include logarithmic terms in the expansion.

Using the linearized Einstein's equations and the Bianchi identities, it can be shown that the perturbed Weyl tensor off of pure AdS-space satisfies the following wave equation:

$$\left(\nabla^e \nabla_e + \frac{2(d-1)}{\ell^2}\right) \delta C_{abcd} = 0. \quad (147)$$

To understand the consequences of this wave equation, we consider the field

$$\Psi \equiv r^{(d-2)/2} Y^{-2} \delta C_{abcd} (\nabla^a Y) t^b (\nabla^c Y) t^d \quad (148)$$

where t, r are the standard global time and radial coordinates of AdS, see eq. (4), $t^a = (\partial/\partial t)^a$, and where $Y = \sqrt{1 + r^2/\ell^2} \cos(t/\ell)$. Thus, at \mathcal{I} , Ψ is essentially the $trtr$ -component of the perturbed Weyl tensor multiplied by a power of r . The other components of the Weyl tensor are addressed in Appendix A with similar results.

Now make a Fourier decomposition of Ψ into modes with time dependence $\exp(-i\omega t/\ell)$, and with angular dependence given by a spherical harmonic on S^{d-2} with angular momentum quantum number l . As we show in Appendix A, from eq. (147), each such mode obeys the ordinary differential equation

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{\nu^2 - 1/4}{\sin^2 x} + \frac{\sigma^2 - 1/4}{\cos^2 x}\right) \Psi = \omega^2 \Psi, \quad (149)$$

where $\sigma = l + (d-3)/2$, $l = 2, 3, \dots$, where $\nu = |d-5|/2$, and where a radial coordinate x has been defined by

$$\frac{\ell}{r} = \tan x, \quad (150)$$

so that $x = 0$ represents points at infinity, while $x = \pi/2$ represents the origin of polar coordinates $r = 0$. We note that the equation satisfied by Ψ has exactly the same form as the ‘‘master equation’’ for the metric perturbations off of pure AdS space found in [43], where a detailed analysis of that equation can be found. The two linearly independent solutions to (149) are given in terms of hypergeometric functions $F = {}_2F_1$. Different combinations of those solutions correspond to different behaviors of Ψ , and hence of the corresponding Weyl tensor component, at infinity $x = 0$ and at the interior point $x = \pi/2$.

With the AdS-CFT correspondence in mind, we restrict attention to solutions that are regular inside the AdS-bulk in the sense that no boundary terms from the interior region of AdS appear in action integral. This regularity requires each mode function Ψ to be vanishing and normalizable at the origin²⁵ $x = \pi/2$. Using the fact that $\sigma \geq 1$, one finds that the solutions with this property are given by

$$\Psi \propto (\sin x)^{\nu+1/2} (\cos x)^{\sigma+1/2} F(\zeta_{\nu,\sigma}^{\omega}, \zeta_{\nu,\sigma}^{-\omega}, 1 + \sigma; \cos^2 x), \quad (151)$$

where

$$\zeta_{\nu,\sigma}^{\omega} \equiv \frac{\nu + \sigma + 1 + \omega}{2}. \quad (152)$$

In order to see the asymptotic behavior of this solution near infinity $x = 0$, it is convenient to transform the argument $\cos x$ into $\sin x$ using well-known transformation identities for hypergeometric functions. For the even- d -case, one obtains

$$\begin{aligned} \Psi \propto (\cos x)^{\sigma+1/2} & \left\{ \frac{\Gamma(1 + \sigma)\Gamma(\nu)}{\Gamma(\zeta_{\nu,\sigma}^{\omega})\Gamma(\zeta_{\nu,\sigma}^{-\omega})} (\sin x)^{-\nu+1/2} F(\zeta_{-\nu,\sigma}^{\omega}, \zeta_{-\nu,\sigma}^{-\omega}, 1 - \nu; \sin^2 x) \right. \\ & \left. + \frac{\Gamma(1 + \sigma)\Gamma(-\nu)}{\Gamma(\zeta_{-\nu,\sigma}^{\omega})\Gamma(\zeta_{-\nu,\sigma}^{-\omega})} (\sin x)^{\nu+1/2} F(\zeta_{\nu,\sigma}^{\omega}, \zeta_{\nu,\sigma}^{-\omega}, 1 + \nu; \sin^2 x) \right\}. \quad (153) \end{aligned}$$

For the odd- d -case, if ω is *not* a real value such that

$$\omega = \mp (2m + 1 + \nu + \sigma), \quad m = 0, 1, 2, \dots, \quad (154)$$

we have

$$\begin{aligned} \Psi \propto (\cos x)^{\sigma+1/2} & \left[\frac{\Gamma(1 + \sigma)\Gamma(\nu)}{\Gamma(\zeta_{\nu,\sigma}^{\omega})\Gamma(\zeta_{\nu,\sigma}^{-\omega})} \sum_{k=0}^{\nu-1} \frac{(\zeta_{-\nu,\sigma}^{\omega})_k (\zeta_{-\nu,\sigma}^{-\omega})_k}{k!(1 - \nu)_k} \cdot (\sin x)^{-\nu+1/2+2k} \right. \\ & - \frac{(-1)^\nu \Gamma(1 + \sigma)}{\Gamma(\zeta_{-\nu,\sigma}^{\omega})\Gamma(\zeta_{-\nu,\sigma}^{-\omega})} \sum_{k=0}^{\infty} \frac{(\zeta_{\nu,\sigma}^{\omega})_k (\zeta_{\nu,\sigma}^{-\omega})_k}{k!(\nu + k)!} \cdot (\sin x)^{\nu+1/2+2k} \cdot \left\{ \log(\sin^2 x) \right. \\ & \left. \left. - \psi(k + 1) - \psi(k + \nu + 1) + \psi(\zeta_{\nu,\sigma}^{\omega} + k) + \psi(\zeta_{\nu,\sigma}^{-\omega} + k) \right\} \right]. \quad (155) \end{aligned}$$

²⁵For this regularity, each mode function Ψ itself need not be smooth at $x = \pi/2$. Indeed, for d -odd case, each normalizable mode function Ψ fails to be smooth at $x = \pi/2$. If one considers a timeslice Σ with a single point removed—as considered in some cases of interest, e.g., a conical singularity as a particle in AdS—then the singular solution may be allowed and come to play some role in AdS-CFT correspondence. Also when some matter fields and/or black holes exist inside the bulk so that non-linear effects become essential, the situation may change.

where $(\zeta)_k \equiv \Gamma(\zeta + k)/\Gamma(\zeta)$ and ψ is the di-gamma function. Note that for the $d = 5$ case, $\nu = 0$ and the first term on the right-hand side of eq. (155) should be ignored.

From these expressions, we immediately see that when $0 \leq \nu < 1$, i.e. in the cases $d = 4, 5, 6$, Ψ is normalizable at $x = 0$ for any $\omega \in \mathbb{C}$. In this case, expanding the above expressions (eq. (153) with $\nu = 1/2$ and eq. (155) with $\nu = 0$) in $\sin x$, we find that for $d = 4, 6$,

$$\Psi = A + B \sin x + C(\sin x)^2 + \dots \quad (156)$$

and for $d = 5$,

$$\Psi = (\sin x)^{1/2}[A \log(\sin x) + B][1 + C(\sin x)^2 + \dots], \quad (157)$$

where ‘ \dots ’ represents a convergent power series in $\sin x$ and the coefficients A, B, \dots depend on the parameters ν, σ , and ω . In particular, the ratio B/A , which takes an arbitrary real number, specifies all possible boundary conditions that can be imposed at $x = 0$ [43].

In the case when $d \geq 7$, the solution Ψ is not normalizable at $x = 0$ when ω fails to satisfy the ‘‘quantization condition’’ eq. (154). On the other hand, when the quantization condition is satisfied, then one can determine²⁶ the asymptotic behavior at $x = 0$ by directly examining the solution, eq. (151), instead of using the expressions (153) and (155). In fact, the quantization condition (154) implies $\zeta_{\nu,\sigma}^\omega = -m$ (or otherwise $\zeta_{\nu,\sigma}^{-\omega} = -m$), hence $F(\zeta_{\nu,\sigma}^\omega, \zeta_{\nu,\sigma}^{-\omega}, 1 + \sigma; \cos^2 x)$ becomes a polynomial of $\cos x$ of order at most $2m$. Therefore the asymptotic behavior of Ψ is simply given by the factor $(\sin x)^{\nu+1/2}$, i.e.,

$$\Psi = (\sin x)^{(d-4)/2} [1 + C \sin x + \dots] \quad (d \geq 7). \quad (158)$$

To see explicitly how the behavior of Ψ which we have just found relates to the behavior of the corresponding Weyl component at \mathcal{I} , we choose the conformal factor as $\Omega = \sin x$. Then Ω^2 times the exact AdS metric is smooth at \mathcal{I} , and

$$\delta \tilde{E}_{ab} t^a t^b \sim \Omega^{-(d-4)/2} \Psi, \quad \text{as } \Omega \rightarrow 0. \quad (159)$$

²⁶When eq. (154) holds, either $\zeta_{\nu,\sigma}^\omega$ or $\zeta_{\nu,\sigma}^{-\omega}$ becomes zero or a negative integer, $-m$, and hence the first term on the right-hand side of eq. (153) vanishes. In particular, for $\nu \in \mathbb{N}$, i.e., $d = 7, 9, 11, \dots$, $\zeta_{\nu,\sigma}^\omega$ (or $\zeta_{\nu,\sigma}^{-\omega}$) also is a negative integer, $-m - \nu$, hence the expression, eq. (155), itself becomes trivial and does not make sense.

The asymptotic formulae eqs. (156), (157), and (158), translate into the following asymptotic behavior of the perturbed electric Weyl tensor:

$$\begin{array}{ll}
d & \delta\tilde{E}_{ab}t^at^b \\
4 & A + B\Omega + \dots \\
5 & (A \log \Omega + B)(1 + C\Omega + \dots) \\
6 & A\Omega^{-1} + B + C\Omega + \dots \\
\geq 7 & 1 + C\Omega^2 + \dots
\end{array} \tag{160}$$

Thus, we arrive at the following conclusion: If we demand normalizability of the perturbed Weyl tensor in the interior, then in $d = 4$ and in $d \geq 7$, the perturbed electric Weyl tensor component $\delta\tilde{E}_{ab}t^at^b$ is smooth at \mathcal{I} . This justifies our boundary condition for asymptotic AdS-spacetime within linear perturbation analysis in these dimensions. In $d = 5$, it is smooth if $A = 0$, but it is logarithmically divergent if $A \neq 0$, while in $d = 6$, it is smooth if $A = 0$ and diverges as Ω^{-1} if $A \neq 0$.

As mentioned above, in $d = 4, 5, 6$, the ratio B/A (with $\pm\infty$ allowed) corresponds to a specific choice of boundary conditions for the wave equation (147). In $d = 4$ there is hence a one-parameter family of boundary conditions giving rise to a smooth perturbed electric Weyl tensor at \mathcal{I} . On the other hand, in $d = 5, 6$, there is only one choice ($A = 0$, essentially ‘‘Dirichlet conditions’’) corresponding to a smooth electric Weyl tensor. The freedom of choosing boundary conditions in the cases $d = 4, 5, 6$ suggests that there are other notions of asymptotically AdS spacetimes in those cases, with consequently different expression for conserved quantities associated with the asymptotic symmetries. On the other hand, in spacetime dimension $d \geq 7$, the boundary conditions are unique, at least at the level of linear perturbation theory.

Actually, as shown in Appendix A, the other electric-type components of the perturbed Weyl tensor, such as $\delta\tilde{E}_{ab}t^a(\partial/\partial\theta^i)^b$, $\delta\tilde{E}_{ab}(\partial/\partial\theta^i)^a(\partial/\partial\theta^j)^b$, also have the same asymptotic behavior as $\delta\tilde{E}_{ab}t^at^b$. Furthermore, all other components are determined from the electric components by the symmetries of the Weyl tensor, the Bianchi identity, and Einstein’s equation. As a result, the above discussion of boundary conditions and the linearized regularity problem of the conformal AdS-infinity \mathcal{I} applies to all types of the Weyl curvature perturbations.

7 Summary

In this work we have compared various constructions of conserved charges (e.g., energy) in asymptotically anti de-Sitter spaces, and also introduced a new construction following the covariant phase space method of Wald et al. [24, 25]. Our main results are as follows:

- In $d \geq 4$ spacetime dimensions the Ashtekar et al definition [5, 6] based on the electric part of the Weyl tensor agrees with the Hamiltonian construction due to Henneaux and Teitelboim [7] and with the covariant phase space definition under suitable asymptotic conditions. These conditions are stated in detail in section 2.
- This agreement occurs because our asymptotic conditions guarantee that the expansion of any asymptotically AdS metric near infinity can be written in the simple form (61,62) in terms of the electric part of the Weyl tensor.
- In $d = 5$ the above definitions of conserved charges differ from the charges defined by the counter-term subtraction method [14, 15]. However, this difference is a function only of the auxiliary conformal structure at \mathcal{I} required by the counter-term subtraction method. This agrees with previous results (e.g., [14, 15, 16, 17, 18, 19, 20, 21, 22, 23]) which evaluated this difference on particular solutions (e.g., Schwarzschild-AdS). Note that, as a result, the counter-term energy is also consistent with the covariant phase space methods of [24], which controls only variations of the Hamiltonian on the space of solutions. Equation (87) displays the explicit formula giving this difference for a general conformal structure. Because this difference is constant over phase space, it has trivial Poisson bracket; that is, the charges generate the same Hamiltonian vector fields. The choice to study the $d = 5$ was made for simplicity. See [35] for an argument based on the Peierls bracket to the effect that a similar conclusion must hold in all dimensions.
- A linearized analysis near infinity and consideration of the resulting back-reaction indicates that our asymptotic conditions are consistent with the dynamics of the metric, of anti-symmetric tensor fields and vector fields with $m^2 \geq 0$, and of scalar fields with m^2 either above or saturating the Breitenlohner-Friedman bound.

We should also emphasize those points not investigated here. For example, our analysis was confined to cases with spacetime dimension $d \geq 4$. The $d = 3$ case will require special treatment due to the frequent appearance of factors of $d - 3$ in section 3. Furthermore, in $d = 3$ spacetime dimensions the Weyl tensor vanishes identically so that the Ashtekar et al definition becomes trivial, though one would still expect the other constructions to agree in the manner described above.

One could also weaken the definition of asymptotically anti-de Sitter in several ways. For example, in the AdS/CFT context it is common to consider spacetime satisfying asymptotic conditions similar to those stated in section 2, but with \tilde{h}_{ab} on \mathcal{I} *not* in the equivalence class of the Einstein static universe. It would be interesting to extend our analysis to this case.

Perhaps the most interesting generalization, however, would be to weaken the asymptotic conditions at infinity and study cases in which the unphysical metric is less smooth. While a linearized analysis indicates that our conditions are consistent with the dynamics of various fields, it also indicates that other consistent conditions should exist. The issue is similar to the use of Dirichlet, Neumann, or another member of the S^1 of consistent linear boundary conditions for the wave equation on the half-line. However, an interesting difference here is that some choices of asymptotic behaviors suggested by the linear analysis allow the fields to grow at infinity so that the linear approximation breaks down.

Some progress has recently been made in constructing conserved quantities under such weakened boundary conditions for scalar fields with masses near the Breitenlohner-Freedman bound [44, 45, 46, 47, 49, 50, 51] using the Henneaux-Teitelboim construction. However, we note that in $d = 4, 5, 6$ the linearized analysis suggests that even for the case of pure Einstein-Hilbert gravity one may be able to define a consistent dynamics (and perhaps conserved charges) with more general asymptotic conditions. This would be interesting to explore in future work.

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Appendix

A Perturbation equations for the Weyl tensor in pure AdS

In this section, we give a derivation of eq. (149) for the Weyl component Ψ . In fact, we will also give a derivation of corresponding equations in static charts other than the global chart used in the main part of the paper, namely the chart with hyperbolic sections, and the chart with flat sections (the ‘‘horospherical chart’’). The latter corresponds to the commonly used ‘‘Poincare coordinates.’’ Some of the material in this section may be of interest more generally, so we give some detail.

A.1 Static charts and background quantities

The metric g_{ab} of pure AdS-space can be written as

$$ds_0^2 = -V^2 dt^2 + \frac{dr^2}{V^2} + r^2 d\sigma_{(K)}^2, \quad (161)$$

where $V = V(r)$ is given as

$$V^2 = K + \frac{r^2}{\ell^2}, \quad (162)$$

and $d\sigma_{(K)}^2$ is the metric of a $(d-2)$ -dimensional constant curvature space with the unit sectional curvature $K = \pm 1, 0$. For the global chart $K = 1$, for the horospherical chart $K = 0$, and for the hyperbolic chart $K = -1$. Note that the horospherical and hyperbolic charts only cover a portion of AdS space, and also that the Killing orbits of $(\partial/\partial t)^a$ are different among different charts. The domains covered by the above charts naturally have the structure of a warped product of an auxiliary 2-dimensional AdS space corresponding to

the r - t directions and the $(d-2)$ -dimensional constant curvature space with $K = \pm 1, 0$, respectively.

To simplify our notation, we denote by $(y^A, \theta^i) = x^\mu$ the coordinates adapted to this product structure, i.e.,

$$g_{AB}dy^A dy^B = -V^2 dt^2 + \frac{dr^2}{V^2}, \quad d\sigma_{(K)}^2 = \sigma_{ij}d\theta^i d\theta^j. \quad (163)$$

Thus, the uppercase roman letters in the range A, B, \dots denote coordinates of the 2-dimensional AdS-space and the lowercase roman letters in the range i, j, \dots denote the angular coordinates θ^i of the $(d-2)$ -dimensional space.

Let D_A and \hat{D}_j be the derivative operators associated, respectively, with g_{AB} and σ_{ij} . Then the derivative operator ∇_a associated with g_{ab} is decomposed using the following formulas for the Christoffel symbols:

$$\Gamma^A_{ij} = -\frac{D^A r}{r}g_{ij} = -rD^A r\sigma_{ij}, \quad \Gamma^i_{Aj} = \frac{D^A r}{r}\delta^i_j, \quad (164)$$

and Γ^A_{BC} and Γ^i_{jk} are identical to the coefficients of D_A and \hat{D}_j , respectively.

Consider a function

$$Y = V(r) \cos(\sqrt{K}t/\ell). \quad (165)$$

Then the gradient $Z_a = \nabla_a Y$ satisfies

$$\nabla_a Z_b = \frac{Y}{\ell^2}g_{ab}. \quad (166)$$

So, Z^a is a homothety vector. For later convenience, we list below some formulas:

$$D_A D_B Z_C = \frac{1}{\ell^2}Z_A g_{BC}, \quad Z_A Z^A = \frac{1}{\ell^2}(Y^2 - K), \quad (167)$$

$$\left(\frac{\dot{Y}}{Y}\right)^2 = \frac{K}{\ell^2}\left(\frac{V^2}{Y^2} - 1\right), \quad \left(V\frac{Y'}{Y}\right)^2 = \frac{r}{\ell^2}, \quad (168)$$

$$\frac{D^A r}{r} = \frac{\ell^2}{r^2}\left(V^2\frac{Z^A}{Y} + \frac{\dot{Y}}{Y}t^A\right), \quad D_C t_A = 2\frac{Z_{[C}t_{A]}}{Y}, \quad (169)$$

$$D_A\left(\frac{Z_C}{Y}\right) = \frac{g_{CA}}{\ell^2} - \frac{Z_C Z_A}{Y^2}, \quad \frac{D_C r}{r}\frac{Z^C}{Y} = \frac{1}{\ell^2}, \quad (170)$$

$$g_{AB} = \frac{\ell^2}{r^2}\left(\frac{K}{Y^2} - 1\right)t^A t^B + V^2\frac{\ell^4}{r^2}\frac{Z_A Z_B}{Y^2} + 2\frac{\ell^4 \dot{Y}}{r^2 Y}\frac{Z_{(A}t_{B)}}{Y}, \quad (171)$$

where the *dot* and the *prime* denote the partial derivatives with respect to t and r , respectively, and $t^a = (\partial/\partial t)^a$ is the background time translation Killing vector.

A.2 Weyl curvature perturbations

Let δC_{abcd} denote perturbations of Weyl curvature on pure AdS-spacetime. Using the Bianchi identity and the Einstein equations, we have

$$\left(\nabla^e \nabla_e + \frac{2(d-1)}{\ell^2} \right) \delta C_{abcd} = 0, \quad (172)$$

$$\nabla^a \delta C_{abcd} = 0. \quad (173)$$

We note that δC_{abcd} itself is gauge-invariant since the background Weyl curvatures are all vanishing.

Define

$$\mathcal{E}_{ab} = \delta C_{acbd} Z^c Z^d. \quad (174)$$

Then, from eqs. (172) and (173) and the homothety property of Z^a one finds that

$$\left(\nabla^c \nabla_c + \frac{2(d-2)}{\ell^2} \right) \mathcal{E}_{ab} = 0, \quad (175)$$

$$\nabla^a \mathcal{E}_{ab} = 0. \quad (176)$$

Using the expressions (164) for the Christoffel symbols Γ_{bc}^a , we can decompose the above equations into the following set of evolution equations for the components \mathcal{E}_{AB} , \mathcal{E}_{Ai} and \mathcal{E}_{ij} and constraint equations among them:

$$\begin{aligned} & D^C D_C \mathcal{E}_{AB} + (d-2) \frac{D^C r}{r} D_C \mathcal{E}_{AB} - (d-2) \frac{D^C r}{r} \left(\frac{D_{Ar}}{r} \mathcal{E}_{BC} + \frac{D_{Br}}{r} \mathcal{E}_{CA} \right) \\ & + \frac{2(d-2)}{\ell^2} \mathcal{E}_{AB} + \frac{\hat{\Delta}}{r^2} \mathcal{E}_{AB} \\ & - 2 \frac{D_{Ar}}{r} \hat{D}_m \mathcal{E}^m_B - 2 \frac{D_{Br}}{r} \hat{D}_m \mathcal{E}^m_A + 2 \frac{(D_{Ar}) D_{Br}}{r^2} \mathcal{E}^m_m = 0, \quad (177) \\ & D^C D_C \mathcal{E}_{Aj} + (d-4) \frac{D^C r}{r} D_C \mathcal{E}_{Aj} - \left\{ \frac{D^2 r}{r} + (d-3) \frac{(Dr)^2}{r^2} \right\} \mathcal{E}_{Aj} \end{aligned}$$

$$\begin{aligned}
& -d \frac{(D_{Ar}) D^C r}{r^2} \mathcal{E}_{Cj} + \frac{\hat{\Delta}}{r^2} \mathcal{E}_{Aj} + \frac{2(d-2)}{\ell^2} \mathcal{E}_{Aj} \\
& -2 \frac{D_{Ar}}{r} \hat{D}_m \mathcal{E}^m_j + 2 \frac{D^C r}{r} \hat{D}_j \mathcal{E}_{CA} = 0, \tag{178}
\end{aligned}$$

$$\begin{aligned}
& D^C D_C \mathcal{E}_{ij} + (d-6) \frac{D^C r}{r} D_C \mathcal{E}_{ij} - 2 \left\{ \frac{D^2 r}{r} + (d-4) \frac{(Dr)^2}{r^2} \right\} \mathcal{E}_{ij} + \frac{\hat{\Delta}}{r^2} \mathcal{E}_{ij} + \frac{2(d-2)}{\ell^2} \mathcal{E}_{ij} \\
& + 2 \frac{D^C r}{r} (\hat{D}_i \mathcal{E}_{Cj} + \hat{D}_j \mathcal{E}_{Ci}) + 2 \frac{(D^C r) D_{Ar}}{r^2} \mathcal{E}_{CAgij} = 0, \tag{179}
\end{aligned}$$

and

$$D_C \mathcal{E}^C_A + (d-2) \frac{D^C r}{r} \mathcal{E}^C_A + \frac{D_{Ar}}{r} \mathcal{E}^C_C + \hat{D}_m \mathcal{E}^m_A = 0, \tag{180}$$

$$\hat{D}_m \mathcal{E}^m_j + \frac{1}{r^{d-2}} D_A (r^{d-2} \mathcal{E}^A_j) = 0, \tag{181}$$

where $\hat{\Delta} = \hat{D}^m \hat{D}_m$.

We denote $\mathcal{E} = \mathcal{E}_{AB} t^A t^B$, $\mathcal{E}_i = \mathcal{E}_{Ai} t^A$. Using formulas listed above (167)-(171) and the formulas for \mathcal{E} , \mathcal{E}_i , \mathcal{E}_{ij} below

$$\mathcal{E}_{AB} Z^B = 0, \tag{182}$$

$$Z^A t^B D_C \mathcal{E}_{AB} = -\frac{Y}{\ell^2} \mathcal{E}_{BC} t^B, \tag{183}$$

$$t^A t^B D_C \mathcal{E}_{AB} = D_C \mathcal{E} - 2 \frac{Z_C}{Y} \mathcal{E}, \tag{184}$$

$$Z^A D_C \mathcal{E}_{Aj} = -\frac{Y}{\ell^2} \mathcal{E}_{Cj}, \tag{185}$$

$$t^A D_C \mathcal{E}_{Aj} = D_C \mathcal{E}_j - \frac{Z_C}{Y} \mathcal{E}_j, \tag{186}$$

we have evolution equations for \mathcal{E} , \mathcal{E}_i , \mathcal{E}_{ij}

$$\left[D^A D_A + \left((d-2) \frac{D^C r}{r} - 4 \frac{Z^C}{Y} \right) D_C - \frac{6}{\ell^2} \frac{K}{Y^2} + \frac{1}{r^2} \hat{\Delta} \right] \mathcal{E} = 0, \tag{187}$$

$$\begin{aligned}
& \left[D^A D_A + \left\{ (d-4) \frac{D^C r}{r} - 2 \frac{Z^C}{Y} \right\} D_C - \frac{2}{\ell^2} \frac{K}{Y^2} - (d-3) \frac{K}{r^2} + \frac{1}{r^2} \hat{\Delta} \right] \mathcal{E}_j \\
& = -2 \frac{\ell^2 \dot{Y}}{r^2 Y} \hat{D}_j \mathcal{E}, \tag{188}
\end{aligned}$$

$$\begin{aligned}
& \left[D^A D_A + (d-6) \frac{D^C r}{r} D_C - 2(d-4) \frac{K}{r^2} + \frac{1}{r^2} \hat{\Delta} \right] \mathcal{E}_{ij} \\
& = -2 \frac{\ell^2 \dot{Y}}{r^2 Y} (\hat{D}_i \mathcal{E}_j + \hat{D}_j \mathcal{E}_i) - 2 \left(\frac{\ell^2 \dot{Y}}{r^2 Y} \right)^2 g_{ij} \mathcal{E}. \tag{189}
\end{aligned}$$

From the constraint equations, we obtain

$$\frac{1}{V^2} \frac{r^2}{\ell^2} \left(-t^C + \ell^2 \frac{\dot{Y}}{Y} \frac{D^C r}{r} \right) D_C \mathcal{E} + (d-3) \frac{\dot{Y}}{Y} \mathcal{E} + \frac{1}{\ell^2} \hat{D}^m \mathcal{E}_m = 0, \tag{190}$$

$$\frac{1}{V^2} \frac{r^2}{\ell^2} \left(-t^C + \ell^2 \frac{\dot{Y}}{Y} \frac{D^C r}{r} \right) D_C \mathcal{E}_j + (d-3) \frac{\dot{Y}}{Y} \mathcal{E}_j + \frac{1}{\ell^2} \hat{D}^m \mathcal{E}_{mj} = 0 \tag{191}$$

A.3 Master equation

Although the equations for \mathcal{E} , \mathcal{E}_i , \mathcal{E}_{ij} obtained above are coupled each other, utilizing the symmetry of the $(d-2)$ -dimensional constant curvature space, we can obtain a set of decoupled equations. To do this, we decompose \mathcal{E}_i into a scalar component and a divergence-free vector component with respect to $d\sigma_K^2$, and similarly, decompose \mathcal{E}_{ij} into a scalar, a divergence-free vector, and a transverse-traceless tensor component with respect to $d\sigma_K^2$. It is convenient to introduce harmonic functions $\mathbb{S}_{\mathbf{k}}$, vectors $\mathbb{V}_{\mathbf{k}i}$, and symmetric tensors $\mathbb{T}_{\mathbf{k}ij}$ on σ_{ijK} defined by the eigenvalue equations,

$$(\hat{\Delta} + \mathbf{k}_S^2) \mathbb{S}_{\mathbf{k}} = 0, \tag{192}$$

$$(\hat{\Delta} + \mathbf{k}_V^2) \mathbb{V}_{\mathbf{k}i} = 0, \quad \hat{D}_i \mathbb{V}_{\mathbf{k}}^i = 0, \tag{193}$$

$$(\hat{\Delta} + \mathbf{k}_T^2) \mathbb{T}_{\mathbf{k}ij} = 0, \quad \mathbb{T}_{\mathbf{k}}^i{}_i = 0, \quad \hat{D}_i \mathbb{T}_{\mathbf{k}}^i{}_j = 0. \tag{194}$$

Some properties of these tensor harmonics and decomposition theorems are given in [43, 61]. We then expand \mathcal{E} , \mathcal{E}_i , \mathcal{E}_{ij} in terms of these harmonics as follows:

$$\mathcal{E} = \psi_S \mathbb{S}, \tag{195}$$

$$\mathcal{E}_i = \phi_S \hat{D}_i \mathbb{S} + \psi_V \mathbb{V}_i, \tag{196}$$

$$\mathcal{E}_{ij} = \mathcal{E}_L \sigma_{ij} \mathbb{S} + \mathcal{E}_T \left(\hat{D}_i \hat{D}_j - \frac{1}{d-2} \hat{\Delta} \sigma_{ij} \right) \mathbb{S} + \mathcal{E}_V \hat{D}_{(i} \mathbb{V}_{j)} + \psi_T \mathbb{T}_{ij}, \tag{197}$$

where here and hereafter we omit the suffix \mathbf{k} labeling the eigenvalue of each mode and the mode summation symbol $\sum_{\mathbf{k}}$ over them. We call $(\psi_S, \phi_S, \mathcal{E}_L, \mathcal{E}_T)$ the scalar-type components, (ψ_V, \mathcal{E}_V) the vector-type components, and ψ_T a tensor-type component of the Weyl perturbations. Note that the tensor-type component does not exist for $d = 4$ case.

The expansion coefficients are not independent due to the constraint equations (190) and (191). First, the tracelessness $\mathcal{E}^a_a = 0$ implies that \mathcal{E}_L is described by ψ_S . It follows from eq. (190) that ϕ_S is described in terms of $(\psi_S, \partial_t \psi_S, \partial_r \psi_S)$, and from eq. (191) that \mathcal{E}_T is described in terms of $(\phi_S, \partial_t \phi_S, \partial_r \phi_S)$ and \mathcal{E}_V in terms of $(\psi_V, \partial_t \psi_V, \partial_r \psi_V)$. Therefore once we obtain $\psi_S, \psi_V,$ and ψ_T , we in fact obtain all of the components.

Furthermore, since the tensor harmonics \mathbb{T}_{ij} have $d(d-4)/2$ independent components, the vector harmonics \mathbb{V}_i have $d-3$ independent components, and the scalar harmonics \mathbb{S} have one independent component, one finds the total number of independent components $d(d-3)/2$, which corresponds to the number of dynamical degrees of freedom for gravitational radiation in d -dimensional spacetime. Therefore $\psi_S, \psi_V,$ and ψ_T describe all the dynamical modes of gravitational perturbations.

Now we derive below a master equation that governs $\psi_S, \psi_V, \psi_T,$ and hence all dynamical degrees of freedom of gravitational perturbations. We can express equations (187), (188), (189) as decoupled equations, respectively, for ψ_S, ψ_V, ψ_T :

$$\begin{aligned} & \left(\frac{\partial^2}{\partial t^2} - 4 \frac{\dot{Y}}{Y} \frac{\partial}{\partial t} + 6 \frac{K V^2}{\ell^2 Y^2} \right) \psi_S \\ & = \left\{ \frac{V^2}{r^{d-2}} \frac{\partial}{\partial r} \left(r^{d-2} V^2 \frac{\partial}{\partial r} \right) - 4V^2 \frac{Y'}{Y} V^2 \frac{\partial}{\partial r} - \frac{V^2}{r^2} \mathbf{k}_S^2 \right\} \psi_S, \end{aligned} \quad (198)$$

$$\begin{aligned} & \left(\frac{\partial^2}{\partial t^2} - 2 \frac{\dot{Y}}{Y} \frac{\partial}{\partial t} + 2 \frac{K V^2}{\ell^2 Y^2} \right) \psi_V = \left[\frac{V^2}{r^{d-4}} \frac{\partial}{\partial r} \left(r^{d-4} V^2 \frac{\partial}{\partial r} \right) \right. \\ & \quad \left. - 2V^2 \frac{Y'}{Y} V^2 \frac{\partial}{\partial r} - \{(d-3)K + \mathbf{k}_V^2\} \frac{V^2}{r^2} \right] \psi_V, \end{aligned} \quad (199)$$

$$\frac{\partial^2}{\partial t^2} \psi_T = \left[\frac{V^2}{r^{d-6}} \frac{\partial}{\partial r} \left(r^{d-6} V^2 \frac{\partial}{\partial r} \right) - \{2(d-4)K + \mathbf{k}_T^2\} \frac{V^2}{r^2} \right] \psi_T. \quad (200)$$

Define master variables $\Psi_S, \Psi_V,$ and Ψ_T by

$$\psi_S = Y^2 r^{-(d-2)/2} \Psi_S, \quad \psi_V = Y r^{-(d-4)/2} \Psi_V, \quad \psi_T = r^{-(d-6)/2} \Psi_T, \quad (201)$$

and introduce the following radial function x ;

$$x = -\frac{1}{\ell} \int \frac{dr}{V}, \quad \text{so that} \quad \frac{r}{\ell} = \frac{\sqrt{K}}{\tan(\sqrt{K}x)}, \quad (202)$$

thus $x = 0$ at infinity. Then, one finds that the three equations for Ψ_S , Ψ_V , Ψ_T are expressed exactly in the same form

$$\ell^2 \frac{\partial^2}{\partial t^2} \Psi = \left\{ \frac{\partial^2}{\partial x^2} - \left(\nu^2 - \frac{1}{4} \right) \frac{K}{\sin^2(\sqrt{K}x)} - \left(\sigma^2 - \frac{1}{4} \right) \frac{1}{\cos^2(\sqrt{K}x)} \right\} \Psi \quad (203)$$

where we have denoted Ψ_S , Ψ_V , and Ψ_T universally by Ψ . Here it is understood that $\sin(\sqrt{K}x)/\sqrt{K} = x$ and $\cos(\sqrt{K}x) = 1$ for $K = 0$. The second term of the right-hand side of eq. (203) becomes relevant near infinity (where $x = 0$) and the parameter ν depends only on the spacetime dimension:

$$\nu^2 - \frac{1}{4} = \frac{(d-4)(d-6)}{4}. \quad (204)$$

Without loss of generality, we take ν to be non-negative,

$$\nu = \frac{|d-5|}{2}. \quad (205)$$

The third term on the right-hand side of eq. (203) stems from to the angular-momentum. The parameter σ depends on the spacetime dimension, the sectional curvature K of $d\sigma^2$ and the mode of perturbations,

$$\sigma^2 - \frac{1}{4} = \frac{(d-2)(d-4)}{4} K + \mathbf{k}_K^2, \quad (206)$$

where $\mathbf{k}_K^2 = \mathbf{k}_S^2$ for the scalar-type, $\mathbf{k}_K^2 = \mathbf{k}_V^2 + K$ for the vector-type, and $\mathbf{k}_K^2 = \mathbf{k}_T^2 + 2K$ for the tensor-type perturbation. In particular, for $K = 1$, the eigenvalues are $\mathbf{k}_S^2 = l(l+d-3)$, $\mathbf{k}_V^2 = \mathbf{k}_S^2 - 1$, $\mathbf{k}_T^2 = \mathbf{k}_S^2 - 2$, hence in this case $\mathbf{k}_K^2 = l(l+d-3)$ for all types of perturbations. Therefore, in this case, one can universally take

$$\sigma = l + \frac{d-3}{2}. \quad (207)$$

Since near infinity $Y \sim r/\ell$, the definition eq. (201) and the master equation (203) imply that in the asymptotic region, all the tensorial types of perturbations behave in the same way

$$\psi_{S,V,T} \sim r^{-(d-6)/2} \Psi. \quad (208)$$

For concreteness, let us examine the master equation (203) in the various charts.

The global chart: In this chart, $K = 1$ and the radial coordinates r and x are related by

$$\frac{r}{\ell} = \frac{\cos x}{\sin x}. \quad (209)$$

The master wave equation (203) becomes

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{\nu^2 - 1/4}{\sin^2 x} + \frac{\sigma^2 - 1/4}{\cos^2 x} \right) \Psi = \omega^2 \Psi, \quad (210)$$

where the eigenvalue ω^2 satisfies $\ell^2 \partial^2 \Psi / \partial t^2 = -\omega^2 \Psi$. Note that ω can be an arbitrary complex number if no conditions on the type of solutions are imposed. Note also that in the $K = 1$ case, $\mathbf{k}^2 = l(l + d - 3)$ takes on discrete values, and the same is consequently true for $\sigma = l + (d - 3)/2$, with $l = 2, 3, \dots$. The solutions to this equation are given by hypergeometric functions but the only solution that is regular at the center of $(S^{d-2}, d\sigma_{+1}^2)$ is given by eq. (151).

Horospherical chart: In this chart, $K = 0$ and we have

$$\frac{r}{\ell} = \frac{1}{x}, \quad (211)$$

and the wave equation (203) becomes

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{\nu^2 - 1/4}{x^2} + \mathbf{k}^2 \right) \Psi = \omega^2 \Psi, \quad (212)$$

where ω, \mathbf{k} are real numbers, i.e., in the $K = 0$ case \mathbf{k}^2 is in the continuous spectrum. The solutions to this equation are given in terms of Bessel and Neumann functions, i.e., $\Psi \propto C_1 \sqrt{x} J_\nu(\sqrt{\omega^2 - \mathbf{k}^2} x) + C_2 \sqrt{x} N_\nu(\sqrt{\omega^2 - \mathbf{k}^2} x)$.

Hyperbolic chart: In this chart, $K = -1$ and we have

$$\frac{r}{\ell} = \frac{\cosh x}{\sinh x}. \quad (213)$$

In this chart, there exists an event horizon with respect to the Killing orbits of $(\partial/\partial t)^a$ at $x \rightarrow \infty$. In particular, if one considers a quotient of the background AdS-space by a certain discrete subgroup of the hyperbolic isometries of $d\sigma_{-1}^2$, the resultant spacetime describes a BTZ-type black hole spacetime. The wave equation (203) becomes

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{\nu^2 - 1/4}{\sinh^2 x} + \frac{\sigma^2 - 1/4}{\cosh^2 x} \right) \Psi = \omega^2 \Psi. \quad (214)$$

The solutions to this equation may be given in terms of hypergeometric functions in an analogous manner to the case (210). In this case, however, $\sigma^2 - 1/4$ given by eq. (206) can take a negative value for the higher dimensional case $d \geq 5$, and also $x \rightarrow \infty$ corresponds to the event horizon. Therefore one might need to consider with more care the regularity of solutions at $x \rightarrow \infty$.

B Perturbation equations for anti-symmetric tensor fields in pure AdS

This appendix analyzes the behavior of anti-symmetric tensor fields $A_{a_1 \dots a_p}$ satisfying equations of motion of the form

$$\nabla^b F_{ba_1 \dots a_p} = -m^2 A_{a_1 \dots a_p}, \quad (215)$$

where $F = dA$ and $p + 1 \leq n$. Here we take the fields to propagate in pure anti-de Sitter space. The kinetic operator on the left-hand side is referred to as the Maxwell operator in [39, 40, 41]. These references show that the linearized anti-symmetric tensor fields that arise in reductions of 10- and 11-dimensional supergravity to AdS_5 and AdS_7 satisfy equations of motion of this type with $m^2 \geq 0$.

It is convenient to take the exterior derivative of (215). In doing so, it is useful to note the following relation:

$$\nabla_{[c_1} \nabla^a \nabla_b A_{c_2 \dots c_{p+1}]} = \nabla_{[c_1} \nabla^a F_{bc_2 \dots c_{p+1}]} = -\nabla_{[b} \nabla^a F_{c_1 c_2 \dots c_{p+1}]}. \quad (216)$$

Thus one finds

$$\begin{aligned} -(p+2)m^2 F_{c_1 c_2 \dots c_{p+1}} &= -\nabla_a \nabla^a F_{c_1 c_2 \dots c_{p+1}} + (p+1) \nabla_{[c_1} \nabla^a F_{a|c_2 \dots c_{p+1}]} \\ &= -\nabla_a \nabla^a F_{c_1 c_2 \dots c_{p+1}} - m^2 (p+1) F_{c_1 c_2 \dots c_{p+1}}. \end{aligned} \quad (217)$$

That is,

$$\nabla_a \nabla^a F_{c_1 c_2 \dots c_{p+1}} = m^2 F_{c_1 c_2 \dots c_{p+1}}. \quad (218)$$

It is straightforward to work out these equations in detail in the $K = 0$ static chart of equation (161). Introducing $z = 1/r$, we find

$$\begin{aligned} m^2 F_{\hat{c}_1 \dots \hat{c}_{p+1}} &= -z^2 (-\partial_t^2 + \sigma^{ij} \hat{D}_i \hat{D}_j) F_{\hat{c}_1 \dots \hat{c}_{p+1}} \\ &\quad - z(2p+2-d) \partial_z F_{\hat{c}_1 \dots \hat{c}_{p+1}} + (p+1)(p+1-d) F_{\hat{c}_1 \dots \hat{c}_{p+1}}, \end{aligned} \quad (219)$$

where the hats on the indices \hat{c}_i indicate that we have chosen $\hat{c}_i \neq z$. Defining $F_{\hat{c}_1 \dots \hat{c}_{p+1}} = z^\alpha \chi$ with $\alpha = (-1+d-2p)/2$ and Fourier transforming on \mathcal{S} then places (219) in the standard form of Bessel's equation. Thus one finds that the two independent solutions are Bessel functions whose leading behavior at small z is z^ν with

$$\nu = \pm \sqrt{m^2 + \frac{d(d+1)}{4} - \frac{1+p}{2}}. \quad (220)$$

Since $d \geq p+1$ and $d \geq 3$, the argument of this square root is bounded below by $m^2 + 3/2$. From (215), we then see that the z -components of F must fall-off at least as fast.

We now require (in analogy with [43]) that boundary conditions be imposed such that the time-translation operator is self-adjoint with respect to the natural inner product

$$\int_{\Sigma} \sqrt{g_{\Sigma}} V^{-1} g^{a_1 b_1} \dots g^{a_{p+1} b_{p+1}} F_{a_1 \dots a_{p+1}} F_{b_1 \dots b_{p+1}}, \quad (221)$$

where Σ is a Cauchy surface and $V^2 = z^{-2}$ is the norm of the time translation ∂_t . This will be the case if eigenfunctions of ∂_t satisfying the boundary condition are normalizable in the inner product (221). Since we are interested in the case $m^2 \geq 0$, we have $|\nu| \geq 1$ which requires us to choose the positive branch of (220). In particular, we find $\nu \geq \sqrt{3/2}$. As a result, one may show that the anti-symmetric tensor fields fall off too fast at infinity to contribute to any expression for the conserved energy.

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