

Uniqueness theorem for 5-dimensional black holes with two axial Killing fields

Stefan Hollands^{1*}, Stoytcho Yazadjiev^{1,2†},

¹*Institut für Theoretische Physik, Universität Göttingen,
D-37077 Göttingen, Germany,*

²*Department of Theoretical Physics, Faculty of Physics, Sofia University
5 J. Bourchier Blvd., Sofia 1164, Bulgaria*

Abstract

We show that two stationary, asymptotically flat vacuum black holes in 5 dimensions with two commuting axial symmetries are identical if and only if their masses, angular momenta, and their “rod structures” coincide. We also show that the horizon must be topologically either a 3-sphere, a ring, or a Lens-space. Our argument is a generalization of constructions of Morisawa and Ida (based in turn on key work of Maison) who considered the spherical case, combined with basic arguments concerning the nature of the factor manifold of symmetry orbits.

1 Introduction

A key theorem about 4-dimensional stationary asymptotically flat black holes is that they are uniquely determined by their conserved asymptotic charges—the mass and angular momentum in the vacuum case [3, 31], and the mass, angular momentum and charge in the Einstein-Maxwell case [27, 2]. But the corresponding statement is no longer true in higher dimensions; there are different vacuum solutions with the same mass, and angular momenta [29, 6]. Nevertheless, it is an interesting open question whether an analogous statement might still hold true if a finite number of suitable further parameters associated with the solution is specified in addition to the mass and angular momenta. The purpose of this note is to show that this is indeed true in the special case of stationary, asymptotically flat vacuum black holes in 5 dimensions which have 2 commuting axial¹ symmetries with the property that the exterior

*hollands@theorie.physik.uni-goe.de

†yazadj@theorie.physik.uni-goe.de

¹By this we mean Killing fields whose orbits are periodic. In higher dimensions, the set of fixed points of such a symmetry is actually generically a higher-dimensional “plane”, rather than an “axis.” We nevertheless refer to the symmetries as axial, by analogy to the 4-dimensional case.

of the spacetime contains no points whose isotropy group is discrete. All exact solutions found so far fall in this class.

In fact, what we will show is that the solution is uniquely determined in terms of its mass, the two angular momenta, and a datum called “rod structure” that has been introduced in a somewhat different form from a more local perspective by Harmark [14, 13], see [7] for a special case². The rod structure encodes information about the relative position of the various axis and the horizon, and gives a measure of their lengths. Actually, as we also show, the rod structure in particular determines the topology of the horizon, which we show may be either be a 3-sphere S^3 , a ring $S^2 \times S^1$, or a Lens-space $L(p, q)$. Our proof of these statements uses a known σ -model formulation of the reduced Einstein equations in 5 dimensions due to Maison [25], which is analogous to a formulation previously found by Mazur [27] and used in his uniqueness proof in 4 dimensions. We combine this technique with an elementary analysis of the global structure of the orbit space of the symmetries. Our result generalizes a result of [26] for the special case of S^3 -horizon topology, which has a particularly simple rod structure.

In 5 dimensions, it is not known whether an arbitrary stationary, asymptotically flat vacuum black hole solution will have two commuting axial Killing fields as we are assuming. In fact, the higher dimensional rigidity theorem [18] only guarantees the existence of one axial Killing field in such spacetimes in addition to the timelike Killing field. In this regard, the situation in 5 dimensions is very different from the analogous situation in 4 dimensions: Here the original rigidity theorem [15, 16, 5, 30, 28, 8] also guarantees the existence of one axial Killing field. But this suffices in 4 dimensions to reduce the Einstein equation to the 2-dimensional σ -model equations [27], and this formulation may then be used to prove the uniqueness. By contrast, in 5 dimensions, two axial Killing fields are required to make the analogous argument. As we have said, however, only one axial Killing field appears to be generic.

Our *conventions and notations* follow those of Wald’s textbook [33].

2 Stationary vacuum black holes in n dimensions

Let (M, g_{ab}) be an n -dimensional, analytic, asymptotically flat, stationary black hole spacetime satisfying the vacuum Einstein equations $R_{ab} = 0$, where $n \geq 4$. Let t^a be the asymptotically timelike Killing field, $\mathcal{L}_t g_{ab} = 0$, which we assume is normalized so that $\lim g_{ab} t^a t^b = -1$ near infinity. We denote by $H = \partial B$ the horizon of the black hole, $B = M \setminus I^-(j^+)$, with j^\pm the null-infinities of the spacetime, which are of topology $\mathbb{R} \times \Sigma_\infty$, with Σ_∞ a compact manifold of dimension $n - 2$.³ We assume that H is non-degenerate and that the horizon cross section is a compact connected manifold of dimension $n - 2$. Under these conditions, one of the following 2 statements is true: (i) If t^a is tangent to the null generators of H then the spacetime must be static [32]. (ii) If t^a is not tangent to the null generators of H , then the higher dimensional rigidity theorem [18] states that there exist N additional linear independent, mutually commuting Killing fields $\psi_1^a, \dots, \psi_N^a$, where N is at least equal to 1. These

²As we will explain, we also obtain new constraints on the rod structure that were not obtained in [14, 13].

³In 4 dimensions, Σ_∞ may be shown to be an S^2 under suitably strong additional hypothesis. A discussion of the structure of null-infinity in higher dimensions is given in [19].

Killing fields generate periodic, commuting flows (with period 2π), and there exists a linear combination

$$K^a = t^a + \Omega_1 \psi_1^a + \cdots + \Omega_N \psi_N^a, \quad \Omega_i \in \mathbb{R} \quad (1)$$

so that the Killing field K^a is tangent and normal to the null generators of the horizon H , and

$$K_a \psi_i^a = 0 \quad \text{on } H. \quad (2)$$

Thus, in case (ii), the spacetime is axisymmetric, with isometry group $\mathcal{G} = \mathbb{R} \times U(1)^N$. From K^a , one may define the surface gravity of the black hole by $\kappa^2 = \lim_H (\nabla_a f) \nabla^a f / f$, with $f = (\nabla^a K^b) \nabla_a K_b$ the norm, and it may be shown that κ is constant on H [33]. In fact, the non-degeneracy condition implies $\kappa > 0$.

In case (i), one can prove that the spacetime is actually unique, and in fact isometric to the Schwarzschild spacetime [22] when $n = 4$, for higher dimensions see [12]. In this paper, we will be concerned with case (ii). We restrict attention to the exterior of the black hole, $I^-(g^+)$, which we shall again denote by M for simplicity. We assume that the exterior M is globally hyperbolic. By the topological censorship theorem [9], the exterior M is a simply connected manifold (with boundary $\partial M = H$). To understand better the nature of the solutions, it is useful to bring the field equations into a form that exploits the symmetries of the spacetime. For this, one considers first the factor space $\hat{M} = M/\mathcal{G}$, where \mathcal{G} is the isometry group of the spacetime generated by the Killing fields. Since the Killing fields ψ_i^a in general have zeros, the factor space $\hat{M} = M/\mathcal{G}$ will normally have singularities. We will analyze the manifold \hat{M} in detail in the next section for the case $n = 5, N = 2$.

The full Einstein equations $R_{ab} = 0$ on M imply a set of coupled differential equations for the metric on the open subsets (of dimension $d = n - N - 1$) of the factor space \hat{M} corresponding to points in M that have a trivial isotropy subgroup⁴. To understand these equations in a geometrical way, we note that the projection $\pi : M \rightarrow M/\mathcal{G} = \hat{M}$ defines a \mathcal{G} -principal fibre bundle over these open subsets of \hat{M} (we will call the union of these sets the ‘‘interior’’ of \hat{M}). At each point x in a fibre over $\pi(x)$ in the interior of \hat{M} , we may uniquely decompose the tangent space $T_x M$ into a subspace of vectors tangent to the fibres, and a space H_x of vectors orthogonal to the fibres. Evidently, the distribution of vector spaces H_x is invariant under the group \mathcal{G} of symmetries, and hence forms a ‘‘horizontal bundle’’ in the terminology of principal fibre bundles [24]. According to one of the equivalent definitions of a connection in the theory of principal fibre bundles [24], a horizontal bundle is equivalent to the specification of a \mathcal{G} -gauge connection \hat{D}_a on the factor space, whose curvature we denote by \hat{F}_{ab} . The horizontal bundle gives an isomorphism $H_x \rightarrow T_{\pi(x)} \hat{M}$ for any x , and this isomorphism may be used to uniquely construct a smooth covariant tensor field $\hat{t}_{ab\dots c}$ on the interior of \hat{M} from any smooth \mathcal{G} -invariant covariant tensor field $t_{ab\dots c}$ on M . For example, the metric g_{ab} on M thereby gives rise to a metric \hat{g}_{ab} on \hat{M} . We let \hat{D}_a act on ordinary tensors $\hat{t}_{ab\dots c}$ as the connection of \hat{g}_{ab} , with Ricci tensor denoted \hat{R}_{ab} .

By performing the well-known ‘‘Kaluza-Klein’’ reduction of the metric g_{ab} on M , we can locally write the Einstein equations as a system of equations on the factor space \hat{M} in terms of metric \hat{g}_{ab} , the components $\hat{F}_{Iab}, I = 0, 1, \dots, N$ of the curvature and the $(N + 1) \times (N + 1)$

⁴The isotropy subgroup of a point $x \in M$ is the subgroup $\{g \in \mathcal{G}; g \cdot x = x\}$.

Gram matrix field G_{IJ}

$$G_{IJ} = g_{ab}X_I^a X_J^b, \quad X_I^a = \begin{cases} t^a & \text{if } I = 0, \\ \psi_i^a & \text{if } I = i = 1, \dots, N. \end{cases} \quad (3)$$

The resulting equations are similar in nature to the ‘‘Einstein-equations’’ on \hat{M} for \hat{g}_{ab} , coupled to the ‘‘Maxwell fields’’ \hat{F}_{ab}^I and the ‘‘scalar fields’’ G_{IJ} , see [23, 4]. We will not write these equations down here, as we will not need them in this most general form.

The equations simplify considerably if the distribution of horizontal subspaces H_x is locally integrable, i.e., locally tangent to a family of $(n - N - 1)$ -dimensional submanifolds. In that case, the connection is flat, $\hat{F}_{ab}^I = 0$, and the dimensionally reduced equations may be written as

$$\hat{D}^a(rG^{-1}\hat{D}_a G)_I^J = 0 \quad (4)$$

together with

$$\hat{R}_{ab} = \hat{D}_a \hat{D}_b \log r - \frac{1}{4}(\hat{D}_a G^{-1})^{IJ} \hat{D}_b G_{IJ}. \quad (5)$$

The equations are well-defined at points in the interior of \hat{M} , corresponding to points with trivial isotropy subgroup. At such points, the matrix G is not singular, i.e., the Gram determinant

$$r^2 = |\det G| \quad (6)$$

does not vanish. Conversely, one may find stationary axisymmetric solutions to the Einstein equations by solving the above equations subject to appropriate boundary conditions on \hat{M} which ensure that the metric g_{ab} reconstructed from \hat{g}_{ab} and G_{IJ} is smooth.

Taking the trace of the first equation, one finds that r is a harmonic function on the interior of \hat{M} ,

$$\hat{D}^a \hat{D}_a r = 0. \quad (7)$$

If \hat{M} has the structure of a manifold with boundary (as we will prove in the next section for the situation considered in this paper), then on the boundary of \hat{M} we have $r = 0$. We may divide the boundary into a (i) a part corresponding to H where $r = 0$ by eq. (2), and (ii) a part corresponding to various ‘‘axis,’’ where G_{IJ} has a null space and where consequently one or more linear combinations of the axial Killing fields vanish. For an asymptotically flat spacetime, the quantity r must be approximately equal in an asymptotically Minkowskian coordinate system to the corresponding quantity formed from N commuting axial Killing fields and $\partial/\partial t$ on exact Minkowski spacetime. Thus, in the region of \hat{M} corresponding to a neighborhood of infinity of M , and away from the axis, $r \rightarrow \infty$. By the maximum principle, r must therefore be in the range $0 < r < \infty$ in the interior of \hat{M} . Thus, in this case, the fields $(G^{-1})^{IJ}$ are globally defined on the interior of \hat{M} , and therefore likewise the dimensionally reduced Einstein equations.

3 The factor space \hat{M}

In this section, we analyze in some detail the factor space $\hat{M} = M/G$ in the case when the dimension of M is equal to five. To begin, we consider a somewhat simpler situation in

which we have a Riemannian 4-manifold (Σ, h_{ab}) with an isometry group $\mathcal{K} = U(1) \times U(1)$, which may be thought of as a spatial slice of our spacetime M . We denote the elements of the isometry group by $k = (e^{i\tau_1}, e^{i\tau_2})$ with $0 \leq \tau_1, \tau_2 < 2\pi$, and we denote the Killing vector fields generating the action of the respective $U(1)$ factors by ψ_1^a respectively ψ_2^a , $\mathcal{L}_{\psi_1} h_{ab} = 0 = \mathcal{L}_{\psi_2} h_{ab}$. These vector fields commute,

$$0 = [\psi_1, \psi_2]^a = \psi_1^b D_b \psi_2^a - \psi_2^b D_b \psi_1^a. \quad (8)$$

We denote the action of a symmetry on a point x by $k \cdot x$. As is common, we call the set $O_x = \{k \cdot x \mid k \in \mathcal{K}\}$ the orbit of the point x , and we call $\mathcal{K}_x = \{k \in \mathcal{K} \mid k \cdot x = x\}$ the isotropy subgroup. As part of our technical assumptions, we assume that the action is such that there are no points with a discrete isotropy subgroup. It is elementary to show that if ψ_1^a, ψ_2^a respectively $\tilde{\psi}_1^a, \tilde{\psi}_2^a$ are two pairs of commuting Killing fields generating such an action of \mathcal{K} , then they must be related by a matrix of integers n_j^i ,

$$\tilde{\psi}_i^a = \sum_{j=1}^2 n_i^j \cdot \psi_j^a, \quad \begin{pmatrix} n_1^1 & n_1^2 \\ n_2^1 & n_2^2 \end{pmatrix} \in GL(2, \mathbb{Z}) \Leftrightarrow \det \begin{pmatrix} n_1^1 & n_1^2 \\ n_2^1 & n_2^2 \end{pmatrix} = \pm 1. \quad (9)$$

We denote the Gram matrix of the Killing fields by $f_{ij} = h_{ab} \psi_i^a \psi_j^b$.

The general structure of the orbit space $\hat{\Sigma} = \Sigma / \mathcal{K}$ can be analyzed by elementary means and is described by the following proposition:

Proposition 1: The orbit space $\hat{\Sigma} = \Sigma / \mathcal{K}$ is a 2-dimensional manifold with boundaries and corners, i.e., a manifold locally modelled over $\mathbb{R} \times \mathbb{R}$ (interior points), $\mathbb{R}_+ \times \mathbb{R}$ (1-dimensional boundary segments) and $\mathbb{R}_+ \times \mathbb{R}_+$ (corners). Furthermore, for each of the 1-dimensional boundary segments, the rank of the Gram matrix f_{ij} is precisely 1, and there is a vector $\mathbf{v} = (v^1, v^2)$ with integer entries such that $f_{ij} v^j = 0$ for each point of the segment. If \mathbf{v}_i respectively \mathbf{v}_{i+1} are the vectors associated with two adjacent boundary segments meeting in a corner, then we must have

$$\begin{pmatrix} v_i^1 & v_{i+1}^1 \\ v_i^2 & v_{i+1}^2 \end{pmatrix} \in GL(2, \mathbb{Z}) \Leftrightarrow \det(\mathbf{v}_i, \mathbf{v}_{i+1}) = \pm 1. \quad (10)$$

On the corners, the Gram matrix has rank 0, and in the interior it has rank 2.

Proof: At each point $x \in \Sigma$, let $V_x \subset T_x \Sigma$ be the linear span of the Killing fields at x , which is tangent to O_x , the orbit through x . Thus, the orbit has the same dimension as a manifold as the vector space V_x . We let H_x be the orthogonal complement of V_x . Each point $x \in \Sigma$ must be in precisely one of the sets

- 0) S_0 , the set of all points such that the dimension of V_x is 0.
- 1) S_1 , the set of all points such that the dimension of V_x is 1.
- 2) S_2 , the set of all points such that the dimension of V_x is 2.

The set S_2 is open because it coincides with the set of all points such that the smooth function $\det f$ is different from zero, and the set S_0 is closed because it is the set of all points where

the smooth function $\text{Tr } f$ is zero. Evidently, if a point x is in S_i , then the entire orbit O_x is in S_i , too. We will now show how to construct a coordinate chart in a neighborhood of each orbit O_x by considering the different cases separately.

Case 2: If $x \in S_2$, then the orbit O_x has dimension 2. In that case, the isotropy group of x can be at most discrete. However, this cannot be the case by assumption, so the isotropy group is in fact trivial, and this also holds for points in a sufficiently small open neighborhood of O_x . If we now choose a coordinate system $\{y_1, \dots, y_4\}$ in Σ near x such that $(\partial/\partial y_1)^a$ and $(\partial/\partial y_2)^a$ are transverse to V_x , then the surface of constant $y_1 = 0 = y_2$ meets each orbit precisely once sufficiently near x . Thus, $\{y_3, y_4\}$ furnish the desired coordinate system of $\hat{\Sigma}$ near x , showing that this space can be locally modelled over $\mathbb{R} \times \mathbb{R}$ near x .

Case 1: For a point $x \in S_1$, the orbit O_x is one-dimensional, i.e., a loop, and there exists a linear combination

$$s^a = v^1 \psi_1^a + v^2 \psi_2^a \quad (11)$$

such that s^a vanishes on O_x , or equivalently $f_{ij} v^i = 0$ there. Hence, $k = (e^{iv^1 \tau}, e^{iv^2 \tau}), 0 \leq \tau < 2\pi$ is in the isotropy subgroup \mathcal{K}_x . Since \mathcal{K}_x is a closed subgroup of the compact group $\mathcal{K} = U(1) \times U(1)$, the ratio v^1/v^2 must either be rational or $\mathcal{K}_x = \mathcal{K}$. The latter would mean that we are in fact in case 0, so we may choose v^1, v^2 to be integers with no common divisor. It then follows that both $(e^{2\pi i/v^2}, 1)$ and $(1, e^{2\pi i/v^1})$ are in the isotropy subgroup. Thus, if we follow the loop O_x by acting with $(e^{i\tau}, 1)$ on x , then we are back to x for the first time after $\tau = 2\pi/u^1$, where u^1 is an integer with $|u^1| \geq |v^2|$, and if we likewise follow the loop by acting with $(1, e^{i\tau})$ on x , then we are back for the first time after $\tau = 2\pi/u^2$, where $|u^2| \geq |v^1|$. The same holds for any other point in the orbit O_x .

To show that the orbit space $\hat{\Sigma}$ can be modelled over $\mathbb{R}_+ \times \mathbb{R}$ near O_x , it is useful to construct a special coordinate system $\{y_1, \dots, y_4\}$ near O_x . This coordinate system is designed in such a way that the action of \mathcal{K} takes a particularly simple form. We let $y_4 = u^1 \tau$ be the parameter along the orbit $\tau \mapsto (e^{i\tau}, 1) \cdot x$. The coordinates $\{y_1, y_2, y_3\}$ measure the geodesic distance from the orbit within a suitable tubular neighborhood, and are defined as follows. First, we pick an orthonormal basis (ONB) $\{\tilde{e}_1^a, \tilde{e}_2^a, \tilde{e}_3^a\}$ of H_x and Lie-drag it along the orbit to an ONB at each $x(\tau), 0 \leq \tau < 2\pi/u^1$. In general, the ONB will not return to itself after we have gone through O_x once, i.e., after $\tau = 2\pi/u^1$, but only after we have gone through it u^1 -times. Consequently, by choosing the ONB at x appropriately, we may assume that

$$(e^{2\pi i/u^1}, 1) \cdot \begin{pmatrix} \tilde{e}_1^a \\ \tilde{e}_2^a \\ \tilde{e}_3^a \end{pmatrix}_x = \begin{pmatrix} \cos(2\pi w^1/u^1) & \sin(2\pi w^1/u^1) & 0 \\ \sin(2\pi w^1/u^1) & \cos(2\pi w^1/u^1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{e}_1^a \\ \tilde{e}_2^a \\ \tilde{e}_3^a \end{pmatrix}_x, \quad (12)$$

for some integer w^1 . In order to obtain an ONB of each $H_{x(\tau)}$ varying smoothly as we go around the loop O_x once (including at $\tau = 2\pi/u^1$), we define

$$\begin{pmatrix} e_1^a \\ e_2^a \\ e_3^a \end{pmatrix}_{x(\tau)} = \begin{pmatrix} \cos(-w^1 \tau) & \sin(-w^1 \tau) & 0 \\ \sin(-w^1 \tau) & \cos(-w^1 \tau) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{e}_1^a \\ \tilde{e}_2^a \\ \tilde{e}_3^a \end{pmatrix}_{x(\tau)} \quad (13)$$

i.e., we undo the rotation. We now define a diffeomorphism from the solid tube $B^2 \times B^1 \times S^1$ into an open neighborhood of O_x by

$$(y_1, y_2, y_3, y_4) \mapsto \text{Exp}_{x(\tau)}(y_1 e_1^a + y_2 e_2^a + y_3 e_3^a), \quad (14)$$

where $y_4 = u^1\tau$ is a periodic coordinate with period 2π , and y_1, y_2, y_3 are sufficiently small. (B^2 is a small open disk around the origin in \mathbb{R}^2 and B^1 a small interval around the origin in \mathbb{R}^1 .) This diffeomorphism defines the desired coordinates. By construction, the action of $(e^{i\tau}, 1)$ is given in these coordinates by

$$y_1 \mapsto \cos(w^1\tau)y_1 + \sin(w^1\tau)y_2 \quad (15)$$

$$y_2 \mapsto \cos(w^1\tau)y_2 + \sin(w^1\tau)y_1 \quad (16)$$

$$y_3 \mapsto y_3 \quad (17)$$

$$y_4 \mapsto y_4 + u^1\tau. \quad (18)$$

The action of $(e^{iv^1\tau}, e^{iv^2\tau})$ on these coordinates can be found as follows: First, the action of $(e^{iv^1\tau}, e^{iv^2\tau})$ leaves each point $x(\tau)$ in the orbit O_x invariant, and it also maps each space $H_{x(\tau)}$ to itself. Furthermore, since s^a and hence $D_a s^b$ is invariant under the action of $(e^{i\tau}, 1)$, it follows that the component matrix of $D_a s^b|_{x(\tau)}$ in the ONB $\{e_1^a, e_2^a, e_3^a\}$ commutes with the matrix in eq. (12). Thus, it must be a rotation in the plane spanned by $e_1^a|_{x(\tau)}, e_2^a|_{x(\tau)}$. Therefore, it follows that the action of $(e^{iv^1\tau}, e^{iv^2\tau})$ is given by

$$y_1 \mapsto \cos(N\tau)y_1 + \sin(N\tau)y_2 \quad (19)$$

$$y_2 \mapsto \cos(N\tau)y_2 + \sin(N\tau)y_1 \quad (20)$$

$$y_3 \mapsto y_3 \quad (21)$$

$$y_4 \mapsto y_4 \quad (22)$$

for some integer N . The action of $(1, e^{i\tau})$ on our coordinates may now be determined in the same way, and is given in terms of integers u^2, w^2 by

$$y_1 \mapsto \cos(-w^2\tau)y_1 + \sin(-w^2\tau)y_2 \quad (23)$$

$$y_2 \mapsto \cos(-w^2\tau)y_2 + \sin(-w^2\tau)y_1 \quad (24)$$

$$y_3 \mapsto y_3 \quad (25)$$

$$y_4 \mapsto y_4 - u^2\tau. \quad (26)$$

Our arguments so far can be summarized by saying that $\{y_1, \dots, y_4\}$ furnish a coordinate system covering a tubular neighborhood of the orbit O_x , with y_4 a 2π periodic coordinate system going around the loop O_x once. The Killing fields ψ_1^a, ψ_2^a are given in terms of these coordinates by

$$\begin{pmatrix} \psi_1^a \\ \psi_2^a \end{pmatrix} = \begin{pmatrix} u^1 & w^1 \\ -u^2 & -w^2 \end{pmatrix} \begin{pmatrix} l^a \\ m^a \end{pmatrix}, \quad (27)$$

where the vector fields l^a, m^a generate the longitude respectively the meridian of the tori of constant $R = (y_1^2 + y_2^2)^{1/2}$ and constant y_3 . They are given in terms of the coordinates by

$$l^a = \left(\frac{\partial}{\partial y_4} \right)^a, \quad m^a = y_1 \left(\frac{\partial}{\partial y_2} \right)^a - y_2 \left(\frac{\partial}{\partial y_1} \right)^a. \quad (28)$$

By the remarks at the beginning of this section, since m^a and l^a locally generate an action of \mathcal{X} which has no points with discrete isotropy group, the determinant $u^1 w^2 - u^2 w^1$ must be ± 1 . In view of the definition of s^a , eq. (11), and the fact that $s^a = N m^a$, it also follows that

$$u^1 v^1 - u^2 v^2 = 0, \quad v^1 w^1 - v^2 w^2 = N. \quad (29)$$

The first equation implies that $u^1 = cv^2$ and $u^2 = cv^1$ for some c . Since the modulus of u^1 is bigger or equal than that of v^2 (and the same with 1 and 2 reversed), we must have $|c| \geq 1$. In view of the second equation, this implies that $|N| = |c| = 1$, and hence that $u^1 = v^2, u^2 = v^1$.

The orbit space may now be determined. We have shown that the orbits of $(e^{i\tau}, 1)$ and $(1, e^{i\tau})$ have the structure of a Seifert fibration, times an interval for the coordinate y_3 . The fibrations are characterized by the winding numbers (v^2, w^1) and $(-v^1, -w^2)$ respectively. Thus, for example the first fibration is such that as the l^a generator winds around v^2 -times, the generator m^a winds around w^1 -times, and similarly for the other action. Thus, if we factor by the action of $(e^{i\tau}, 1)$, we locally obtain the space $\mathbb{R} \times (\mathbb{R}^2/\mathbb{Z}_{v^2})$, where $\mathbb{Z}_p \subset U(1)$ is the cyclic subgroup of p elements whose action on $\mathbb{R}^2 \cong \mathbb{C}$ is generated by the phase multiplication $z \mapsto e^{2\pi i/p} z$. The factor \mathbb{R} in the Cartesian product corresponds to the coordinate y_3 , while the other factor to the coordinates y_1, y_2 . We next factor by $(1, e^{i\tau})$. Since the only nontrivial part of this action on $\mathbb{R} \times (\mathbb{R}^2/\mathbb{Z}_{v^2})$ is a rotation in the cone $\mathbb{R}^2/\mathbb{Z}_{v^2}$, we may parametrize the orbits in a neighborhood of O_x by y_3 and $R = (y_1^2 + y_2^2)^{1/2}$. This shows that, in case (1), $\hat{\Sigma}$ locally has the structure $\mathbb{R} \times \mathbb{R}_+$. On the edge locally defined by $R = 0$, we have $v^1 \psi_1^a + v^2 \psi_2^a = 0$.

[If we had first factored by the action of $(1, e^{i\tau})$, we would have locally obtained the space $\mathbb{R} \times (\mathbb{R}^2/\mathbb{Z}_{v^1})$. The rest would be analogous.]

Case 0: If $x \in S_0$, then $\psi_1^a = 0 = \psi_2^a$ at the point x , and the linear transformations $D_a \psi_1^b, D_a \psi_2^b$ in the tangent space $T_x \Sigma$ can be viewed as elements of the Lie-algebra $\mathfrak{o}(4)$ of $O(4)$, defined with respect to the Riemannian metric h_{ab} on $T_x \Sigma$. Taking a derivative of eq. (8) and evaluating at x , it follows that these linear transformations commute at x ,

$$(D_a \psi_1^b) D_b \psi_2^c - (D_a \psi_2^b) D_b \psi_1^c = 0 \quad \text{at } x. \quad (30)$$

This means that, if we form the self-dual and anti-self-dual parts

$$D_a \psi_{1b} \pm \frac{1}{2} \varepsilon_{ab}{}^{cd} D_c \psi_{1d}, \quad D_a \psi_{2b} \pm \frac{1}{2} \varepsilon_{ab}{}^{cd} D_c \psi_{2d}, \quad (31)$$

then the self-dual part of $D_a \psi_{1b}$ must be proportional to that of $D_a \psi_{2b}$ at x , and similarly for the anti-self-dual parts, as one may see using the Lie-algebra isomorphism between $\mathfrak{o}(4)$ and $\mathfrak{o}(3) \times \mathfrak{o}(3)$ corresponding to the decomposition into self-dual and anti-self-dual parts. Now pick an orthonormal tetrad $\{e_1^a, e_2^a, e_3^a, e_4^a\}$ at x . Then basis for the 3-dimensional spaces of self-dual and anti-self-dual skew 2-tensors on $T_x \Sigma$ are given by $e_{1[a} e_{2b]} \pm e_{3[a} e_{4b]}$, $e_{1[a} e_{3b]} \pm e_{2[a} e_{4b]}$ and $e_{1[a} e_{4b]} \pm e_{2[a} e_{3b]}$, respectively. Performing an $O(4)$ rotation of the tetrad corresponds to two independent $O(3)$ -rotations of the respective basis of self-dual and anti-self-dual tensors, and vice versa. It follows that tetrad may be rotated if necessary so that the self-dual parts of $D_a \psi_{1b}$ and $D_a \psi_{2b}$ are proportional $e_{1[a} e_{2b]} + e_{3[a} e_{4b]}$, and the anti-self-dual parts are proportional to $e_{1[a} e_{2b]} - e_{3[a} e_{4b]}$. Therefore we may write, at x ,

$$\begin{pmatrix} D_a \psi_{1b} \\ D_a \psi_{2b} \end{pmatrix} = \begin{pmatrix} n_1^1 & n_1^2 \\ n_2^1 & n_2^2 \end{pmatrix} \begin{pmatrix} 2e_{1[a} e_{2b]} \\ 2e_{3[a} e_{4b]} \end{pmatrix} \quad (32)$$

for some matrix n_j^i . Let us now pick Riemannian normal coordinates $\{y_1, y_2, y_3, y_4\}$ centered at x corresponding to our choice of tetrad. Then, since the Killing fields ψ_1^a and ψ_2^a are

globally determined by the tensors $D_a \psi_1^b$ and $D_a \psi_2^b$ at the point x , it follows from eq. (32) that $\psi_i^a = \sum n_i^j \cdot s_j^a$ in an open neighborhood of x , where

$$s_1^a = y_1 \left(\frac{\partial}{\partial y_2} \right)^a - y_2 \left(\frac{\partial}{\partial y_1} \right)^a, \quad s_2^a = y_3 \left(\frac{\partial}{\partial y_4} \right)^a - y_4 \left(\frac{\partial}{\partial y_3} \right)^a. \quad (33)$$

Since both sets of Killing fields s_i^a and ψ_i^a have periodic orbits with period 2π , both the matrix n_i^j and the matrix $v_i^j = (n^{-1})^i_j$ must be integer valued. We now define $R_1 = (y_1^2 + y_2^2)^{1/2}$, $R_2 = (y_3^2 + y_4^2)^{1/2}$. These quantities are clearly invariant under the action of \mathcal{K} and in 1—1 correspondence with the orbits near O_x . This gives $\hat{\Sigma}$ the structure of $\mathbb{R}_+ \times \mathbb{R}_+$ near the orbit O_x . On the edges locally defined by $R_i = 0$, we have $v_i^1 \psi_1^a + v_i^2 \psi_2^a = 0$.

We have now constructed the desired coordinate systems in the above 3 cases, and it can be checked that the transition functions are smooth. Thus we have shown that $\hat{\Sigma}$ has the structure of a manifold with boundaries and corners. \square

The same technique of proof may be used to analyze the possible horizon topologies of stationary, asymptotically flat black hole spacetimes with an action of $\mathcal{K} = U(1) \times U(1)$ satisfying the hypothesis that there are no points with discrete isotropy group under \mathcal{K} .

Proposition 2: Under the above hypothesis, each connected component of the horizon cross section \mathcal{H} must be topologically either a ring $S^1 \times S^2$, a sphere S^3 , or a Lens-space $L(p, q)$, with $p, q \in \mathbb{Z}$.

Remark 1: The Lens-spaces $L(p, q)$ (see e.g. [1, Paragraph 9.2]) are the spaces obtained by factoring the unit sphere S^3 in \mathbb{C}^2 by the group action $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$. The fundamental group of the Lens space is $\pi_1(L(p, q)) = \mathbb{Z}_p$, and q is determined only up to integer multiples of p . Since a Lens-space is a quotient of the positive constant curvature space S^3 by a group of isometries, it can carry a metric of everywhere positive scalar curvature, like the other possible topologies S^3 and $S^2 \times S^1$. Thus, the possible horizon topologies listed in Proposition 2 are of so-called ‘‘positive Yamabe type,’’ in accordance with a general theorem [11].

Proof: As a result of the rigidity theorem [18], we can find a horizon cross section \mathcal{H} which is itself a Riemannian manifold with induced metric q_{ab} , of dimension 3, invariant under the group $\mathcal{K} = U(1) \times U(1)$ of axial symmetries generated by ψ_1^a and ψ_2^a . By the same arguments as in the proof of Proposition 1, \mathcal{H} divided by \mathcal{K} is a 1-dimensional manifold with boundary, i.e., a union of intervals, each of which corresponds to a connected component of \mathcal{H} . We restrict attention to one connected component of \mathcal{H} , whose space of orbits is a single interval. The end points of the interval correspond to 1-dimensional orbits where a linear combination of the axial Killing fields vanishes⁵. We call these orbits O_{x_1} and O_{x_2} . They are closed loops. All other points of the interval correspond to non-degenerate orbits diffeomorphic to the 2-torus $S^1 \times S^1$. At x_1 , an integer linear combination $m_1^a = v_1^1 \psi_1^a + v_1^2 \psi_2^a$ vanishes, while at x_2 , an integer linear combination $m_2^a = v_2^1 \psi_1^a + v_2^2 \psi_2^a$ vanishes. As in the proof of Proposition 1, we may introduce a local coordinate systems in tubular neighborhoods of O_{x_1}

⁵There cannot be points x in \mathcal{H} where both ψ_1^a and ψ_2^a vanish, since $D_a \psi_1^b$ and $D_a \psi_2^b$ would otherwise be two commuting but not linearly dependent infinitesimal $SO(3)$ rotations in the tangent space of x , which is impossible.

and O_{x_2} such that each neighborhood is diffeomorphic to a solid tube $S^1 \times B^2$. We denote the radial coordinates measuring the distance from the origin in each of the discs B^2 by R_1 for the first tubular neighborhood, and by R_2 for the second tubular neighborhood. By construction, the tori of constant R_1 respectively R_2 correspond to 2-dimensional orbits of \mathcal{K} , i.e., interior points of the interval. In fact, R_1 and R_2 measure the distance of the interior point of the interval to the first respectively second boundary point.

If m_1^a, l_1^a are the meridian of a torus of constant r_1 in the first tubular neighborhood (with the longitude going around the S^1 -direction in the cartesian product $S^1 \times B^2$), and m_2^a, l_2^a the corresponding quantities for the second tubular neighborhood, then as in case 1 in the proof of Proposition 1, we have

$$\begin{pmatrix} \Psi_1^a \\ \Psi_2^a \end{pmatrix} = \begin{pmatrix} v_1^2 & w_1^1 \\ -v_1^1 & -w_1^2 \end{pmatrix} \begin{pmatrix} l_1^a \\ m_1^a \end{pmatrix} = \begin{pmatrix} v_2^2 & w_2^1 \\ -v_2^1 & -w_2^2 \end{pmatrix} \begin{pmatrix} l_2^a \\ m_2^a \end{pmatrix}. \quad (34)$$

We must now smoothly join the coordinate systems defining the tubular neighborhoods of O_{x_1} respectively O_{x_2} . Each tubular neighborhood is a solid torus $B^2 \times S^1$. Their boundaries (each diffeomorphic to a torus $S^1 \times S^1$) must be glued together in such a way that the orbits of Ψ_1^a and Ψ_2^a match. In order to exploit this fact, we act with the inverse of the second matrix on eq. (34), to obtain the relation $m_1^a = pl_2^a + qm_2^a$, where

$$q = w_2^1 v_1^1 - w_2^2 v_1^2, \quad p = v_1^1 v_2^2 - v_1^2 v_2^1 = \det(\mathbf{v}_1, \mathbf{v}_2). \quad (35)$$

This means that, while the meridian goes around the the torus bounding the first tubular neighborhood once, it goes p -times around the longitude and q -times around the meridian of the torus bounding the second tubular neighborhood. These solid tubes have to be glued together accordingly. When $p \neq 0 \neq q$, the manifold thereby obtained is topologically a Lens space $L(p, q)$ according to one of the equivalent definitions of this space. Note that q is defined in terms of the vectors $\mathbf{v}_1, \mathbf{v}_2$ by the above equation up to an integer multiple sp , since the vectors \mathbf{w}_1 respectively, \mathbf{w}_2 are only defined up to integer multiples of \mathbf{v}_1 respectively \mathbf{v}_2 by the condition that the matrices in eq. (34) have determinant ± 1 . However, the Lens $L(p, q)$ and $L(p, q + sp)$ are known to be equivalent, so the Lens space is determined uniquely by the pair $(\mathbf{v}_1, \mathbf{v}_2)$.

If $q = 0$ modulo $p\mathbb{Z}$, then $p = \pm 1$, and vice versa. In that case, we may similarly argue as above and show that \mathcal{H} is topologically S^3 . Finally, if $p = 0$, then $q = \pm 1$ and vice versa, and we may argue as above to show that \mathcal{H} is topologically $S^2 \times S^1$. \square

Remark 2: The proof shows how the different topologies $S^3, S^2 \times S^1, L(p, q)$ are related to the kernel of the Gram matrix $G_{ij} = g_{ab} \Psi_i^a \Psi_j^b$ at the 2 boundary points of the interval $I = \mathcal{H} / \mathcal{K}$, i.e., the ‘‘rod-vectors’’ introduced in the next section: If we denote the integer-valued vectors in the kernel by $\mathbf{v}_1, \mathbf{v}_2$, and set $p = \det(\mathbf{v}_1, \mathbf{v}_2)$, then the topolgy of \mathcal{H} is $S^2 \times S^1$ if $p = 0$, it is S^3 if $p = \pm 1$, and a Lens space $L(p, q)$ otherwise.

We finally consider in detail the orbit space $\hat{M} = M / \mathcal{G}$ of a stationary, asymptotically flat, Lorentzian vacuum black hole spacetime (M, g_{ab}) of dimension 5 with 2-dimensional axial symmetry group $\mathcal{K} = U(1) \times U(1)$. The Killing field t^a that is timelike near infinity corresponds to the isometry group \mathbb{R} , so that the full symmetry group is $\mathcal{G} = \mathcal{K} \times \mathbb{R}$. As

above, we assume that there are no points in the exterior of M whose isotropy subgroup \mathcal{K}_x is discrete. We denote the exterior of the black hole again by M , so that M itself is a manifold with boundary $\partial M = H$. We also assume that M is globally hyperbolic. First, we note that t^a can nowhere be equal to a linear combination of the axial Killing fields. Indeed, letting F_τ be the flow of t^a , if t^a were a linear combination of the axial Killing fields at a point $x \in M$, then the F_τ -orbit through x would either be periodic (for a rational linear combination), or almost periodic (for an irrational linear combination). This would imply that there are closed (or nearly closed) F_τ -orbits. However, consider the intersection S_τ of $\partial J^+(F_\tau(x))$ with \mathcal{J}^+ . Evidently, on the one hand, S_τ must be bounded as τ varies, because the orbits F_τ are periodic, or almost periodic. On the other hand, near \mathcal{J}^+ , the Killing field t^a is timelike, so the sets S_τ are related by a time-translation, and hence cannot be bounded as τ varies. Thus t^a cannot be tangent to a linear combination of the axial Killing fields at any point.

Next, we show that the linear span V_x of ψ_1^a, ψ_2^a is everywhere spacelike. Indeed if there was a linear combination ξ^a of the axial Killing fields that was timelike or null somewhere, then we could consider the timelike or null orbit of ξ^a . This orbit must necessarily have a closure in M that is non-compact, again invoking the global causal structure of M . On the other hand, ξ^a is a linear combination of axial Killing fields, so it must have either periodic or almost periodic orbits and its closure must hence be isometric to a compact factor group of \mathcal{K} , a contradiction.

Thus, we have now learned that V_x is spacelike for all x , and that t^a is transverse to V_x for all x . This can now be used to determine the general structure of the orbit space \hat{M} . To do this, we split the isometry group $\mathcal{G} = \mathcal{K} \times \mathbb{R}$ into the subgroup \mathbb{R} generated by t^a , and the compact subgroup \mathcal{K} generated by the axial Killing fields. Proceeding as in the proof of Proposition 1, we first consider the factor space M/\mathcal{K} . Using that V_x are everywhere spacelike, it now follows that M/\mathcal{K} is a 3-dimensional manifold with boundaries and corners (of dimension 2 and 1 respectively). We then factor in addition by the subgroup \mathbb{R} . Since the action of \mathbb{R} is nowhere tangent to the orbits of \mathcal{K} , the action is free, and we find that $\hat{M} = (M/\mathcal{K})/\mathbb{R}$ is a 2-dimensional manifold with boundaries and corners.

Finally, we know that M is simply connected by the topological censorship theorem [9, 10]. By standard arguments from homotopy theory, because \mathcal{G} is connected, also the factor space \hat{M} has to be simply connected. We summarize our findings in a Proposition:

Proposition 3: Let (M, g_{ab}) be the exterior of a stationary, asymptotically flat, 5-dimensional vacuum black hole spacetime with isometry group $\mathcal{G} = \mathcal{K} \times \mathbb{R}$, as described above. Then the orbit space $\hat{M} = M/\mathcal{G}$ is a simply connected, 2-dimensional manifold with boundaries and corners. Furthermore, in the interior, on the 1-dimensional boundary segments (except the piece corresponding to H), and on the corners, the Gram matrix $G_{ij} = g_{ab}\psi_i^a\psi_j^b$ has rank precisely 2, 1 respectively 0.

4 Classification of 5-dimensional stationary spacetimes

We now consider again the reduced Einstein equations for a stationary black hole spacetime with $n - 3$ commuting axial Killing fields. We assume that the action isometry group \mathcal{K} generated by the axial symmetries is so that there are no points with discrete isotropy group. We also assume in this section that the infinity is metrically and topologically a sphere,

$\Sigma_\infty = S^{n-2}$. Then $n - 3$ commuting axial Killing fields are only possible when $n = 4, 5$ but not for dimensions $n \geq 6$, because the compact part $SO(n - 2)$ of the asymptotic symmetry group admits at most $(n - 2)/2$ mutually commuting generators⁶. When $n = 4$, the rigidity theorem [16, 5, 30, 8] guarantees the existence of at least one more axial Killing field, so that the total number of Killing fields is at least 2. Thus, for $n = 4$ we are always in the situation just described. If $n = 5$, the higher dimensional rigidity theorem [18] also guarantees at least one more axial Killing field, but for a solution with precisely one extra axial Killing field, we would not be in the situation just described if such solutions were to exist. From now on, we take $n = 5$, and we *postulate* that the number of axial Killing fields is $N = 2$. We also assume that the axial symmetries have been defined so as to act like the standard rotations in the 12-plane resp. 34-plane in the asymptotically Minkowskian region.

As explained in Proposition 3 in the last section, in that case the factor space \hat{M} is a simply connected 2-dimensional manifold with boundaries and corners. As in 4 dimensions, one can show using Einstein's equations and Frobenius' theorem that the horizontal subspaces H_x orthogonal to the Killing fields are locally integrable [33], so the metric may be written as

$$g_{ab} = (G^{-1})^{IJ} X_{Ia} X_{Jb} + \pi^* \hat{g}_{ab} \quad (36)$$

away from points where G is singular, where $\pi : M \rightarrow \hat{M} = M/\mathcal{G}$ is the projection. Furthermore, using that $\det G$ is nowhere vanishing in the interior of \hat{M} and negative near infinity, it follows that \hat{g}_{ab} is a metric of signature $(++)$, i.e., a Riemannian metric. The reduced Einstein equations for this metric are given by eqs. (4) and (5).

Since \hat{M} is an (orientable) simply connected 2-dimensional analytic manifold with boundaries and corners, we may map it analytically to the upper complex half plane $\{\zeta \in \mathbb{C}; \text{Im} \zeta > 0\}$ by the Riemann mapping theorem. Furthermore, since r is harmonic, we can introduce a harmonic scalar field z conjugate to r (i.e., $\hat{D}^a z = \hat{\varepsilon}^{ab} \hat{D}_b r$). Since an analytic mapping is conformal we also have $\partial_\zeta \partial_{\bar{\zeta}} r = 0 = \partial_\zeta \partial_{\bar{\zeta}} z$, and from this, together with the boundary condition $r = 0$ for $\text{Im} \zeta = 0$, one can argue that $\zeta = z + ir$ by a simple argument involving the maximum principle [34]. In particular, r and z are globally defined coordinates, and the metric globally takes the form

$$\hat{g}_{ab} = e^{2\nu(r,z)} [(dr)_a (dr)_b + (dz)_a (dz)_b]. \quad (37)$$

Since eq.(4) is invariant under conformal rescalings of \hat{g}_{ab} , and since a 2-dimensional metric is conformally flat, it decouples from eq. (5). In fact, writing the Ricci tensor \hat{R}_{ab} of (37) in terms of ν , one sees that eq. (5) equation may be used to determine ν by a simple integration.

The coordinate scalar fields r, z on \hat{M} are uniquely defined by the above procedure up to a global conformal transformation of the upper half plane, i.e., a fractional transformation of the form

$$\zeta \mapsto \frac{a\zeta + b}{c\zeta + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1, \quad \zeta = z + ir. \quad (38)$$

We will now show how r, z can in fact be uniquely fixed by a suitable condition near infinity, up to a translation of z . For 5-dimensional Minkowski spacetime, the Killing fields $\psi_1^a = (\partial/\partial\phi_1)^a$ and $\psi_2^a = (\partial/\partial\phi_2)^a$ are rotations in the 12-plane and the 34-plane, and the

⁶If we assume a different topology and metric structure of Σ_∞ , such as $\Sigma_\infty = T^{n-2}$, then the spacetime may have $n - 2$ commuting axial Killing fields.

coordinates r, z as constructed above are given in terms of inertial coordinates by $r = R_1 R_2$ and $z = \frac{1}{2}(R_1^2 - R_2^2)$, with $R_1 = \sqrt{x_1^2 + x_2^2}$ and $R_2 = \sqrt{x_3^2 + x_4^2}$, as well as $\phi_1 = \arctan(x_1/x_2)$ and $\phi_2 = \arctan(x_3/x_4)$. The conformal factor is given by $e^{2\nu} = 1/2\sqrt{r^2 + z^2}$. In the general case, we may pick an asymptotically Minkowskian coordinate system and we may define the quantities r, z on the curved, axisymmetric spacetime under consideration so that they are approximately equal near infinity to the expressions in Minkowski spacetime as just given. In particular, we may achieve that

$$e^{2\nu} \rightarrow \frac{1}{2\sqrt{z^2 + r^2}} \quad (39)$$

near infinity, which corresponds to $r \rightarrow \infty$, as z is fixed or to $z \rightarrow \pm\infty$ for $r = 0$. This condition fixes $a = d = 1, c = 0$ and hence leaves only the freedom of shifting z by a constant. Thus, in summary, the Einstein equations are reduced to the two decoupled equations (4) and (5) on the factor manifold $\hat{M} = \{\zeta = z + ir \in \mathbb{C}; \text{Im} \zeta > 0\}$ with metric (37) and a preferred coordinate system (r, z) that is determined up to a translation of z . The function ν is determined by eq. (5), subject to the boundary condition (39).

So far, our construction is similar to well-known constructions leading to the uniqueness theorems in $n = 4$ spacetime dimensions (for a review, see [17]). In fact, the only apparent difference to 4 dimensions is that the matrix field G_{IJ} is a 3×3 field in 5 dimensions, while it is a 2×2 matrix field in 4 dimensions. In particular, all information about the topology of M and the horizon might seem to be lost. In 4 dimensions, the reduced Einstein equations may be used to prove that stationary metrics are unique for fixed mass and angular momentum. On the other hand, it is known that in 5 dimensions, solutions are not uniquely fixed by these parameters, and that there are even different possibilities for the topology of the horizon. Thus, one naturally wonders where those differences are encoded in the above formulation.

To understand this point, we must remember that the 2-dimensional orbit space \hat{M} is a manifold with boundaries and corners by Proposition 3. The line segments of the boundary correspond to the axis (i.e., the sets where a linear combination $\nu^1 \psi_1^a + \nu^2 \psi_2^a$ vanishes), or to the factor space of the horizon, $\hat{H} = H/\mathcal{G}$. The corners—the intersections of the line segments—correspond to points where the axis intersect (i.e., where both Killing fields vanish simultaneously), or to points where the axis intersect the horizon H . In the realization of \hat{M} as the upper complex half plane, the line segments of $\partial\hat{M}$ correspond to intervals

$$(-\infty, z_1), (z_1, z_2), \dots, (z_k, z_{k+1}), (z_{k+1}, \infty) \quad (40)$$

of the real axis forming the boundary of the upper half plane. Evidently, if the horizon is connected as we assume, precisely one interval (z_h, z_{h+1}) corresponds to the horizon. The other intervals correspond to rotation-axis, while the points z_j correspond to the intersection points of the axis, except for the boundary points of the interval (z_h, z_{h+1}) representing the horizon. Above, we argued that the coordinate z is defined in a diffeomorphism invariant way in terms of the solution up to shifts by a constant. Consequently, the k positive real numbers

$$l_1 = z_1 - z_2, \quad l_2 = z_2 - z_3, \quad \dots \quad l_k = z_k - z_{k+1} \quad (41)$$

are invariantly defined, i.e., are the same for any pair of isometric stationary black hole spacetimes of the type we consider. Thus, they may be viewed as global parameters (“mod-

uli”) characterizing the given solution in addition to the mass m and the two angular momenta J_1, J_2 . Furthermore, with each l_j , there is associated a label which is either a vector $\mathbf{v}_j = (v_j^1, v_j^2)$ of integers such that the linear combination $v_j^1 \Psi_1^a + v_j^2 \Psi_2^a$ vanishes, or $\mathbf{v}_h = (0, 0)$ if we are on the horizon. The labels corresponding to the “outmost” intervals $(-\infty, z_1)$ and (z_{k+1}, ∞) must be $(0, 1)$ respectively $(1, 0)$, because this is the case for Minkowski spacetime, and we assume that our solutions are asymptotically flat. Also from Proposition 1, and the Remark 2 following the proof of Proposition 2, we have

$$\frac{\det(\mathbf{v}_j, \mathbf{v}_{j+1}) = \pm 1}{\det(\mathbf{v}_{h-1}, \mathbf{v}_{h+1}) = p} \left| \begin{array}{l} \text{if } (z_{j-1}, z_j) \text{ and } (z_j, z_{j+1}) \text{ are not the horizon} \\ \text{if } (z_h, z_{h+1}) \text{ is the horizon} \end{array} \right.$$

Moreover, $p = 0$ for $\mathcal{H} \cong S^2 \times S^1$, $p = \pm 1$ for $\mathcal{H} \cong S^3$, and $\mathcal{H} \cong L(p, q)$ is a Lens-space for other values of p . The numbers $\{l_j\}$ and the assignment of the labels $\{\mathbf{v}_j\}$ are related to the “rod-structure” of the solution, introduced from a more local perspective in [14]⁷, see also [7] for a special case. We will therefore simply call the data consisting of $\{l_j\}$ and the assignments $\{\mathbf{v}_j\}$ the rod structure as well.

For 4 dimensional black holes, there is only the trivial rod structure $(-\infty, z_1), (z_1, z_2), (z_2, \infty)$, with the middle interval corresponding to the horizon, and the first and third corresponding to single axis of rotation of the Killing field. Furthermore, the rod length l_1 may be expressed in terms of the global parameters m, J of the solution. By contrast, in 5 dimensions, the rod structure can be non-trivial, and in fact differs for the Myers-Perry [29] and Black Ring [6] solutions. For these cases, the rod structure is summarized in the following table [14]:

	Rods	Rod Vectors (Labels)	Horizon Topology
Myers-Perry BH	$(-\infty, z_1), (z_1, z_2), (z_2, \infty)$	$(1, 0), (0, 0), (0, 1)$	S^3
Black Ring	$(-\infty, z_1), (z_1, z_2), (z_2, z_3), (z_3, \infty)$	$(1, 0), (0, 0), (1, 0), (0, 1)$	$S^2 \times S^1$
Flat Spacetime	$(-\infty, z_1), (z_1, \infty)$	$(1, 0), (0, 1)$	—

The following rod structure would represent a “Black Lens” if such a solution would exist:

	Rods	Rod Vectors (Labels)	Horizon Topology
Black Lens	$(-\infty, z_1), (z_1, z_2), (z_2, z_3), (z_3, \infty)$	$(1, 0), (0, 0), (1, n), (0, 1)$	$L(n, 1)$

Even for a fixed set of asymptotic charges m, J_1, J_2 the invariant lengths of the rods $l_1 = z_1 - z_2, l_2 = z_2 - z_3$ may be different for the different Black Ring solutions, corresponding to the fact that there exist non-isometric Black Ring solutions with equal asymptotic charges [6, 7]. On a rod labelled “(1,0)”, all components of $G_{1j} = G_{j1}, j = 1, 2$ vanish but not the other ones, while on a rod labelled “(0,1)”, all components $G_{2j} = G_{j2}$ vanish. The vector $(1, 0)$ hence corresponds to a $\partial/\partial\phi_1$ -axis, while the vector $(0, 1)$ corresponds to a $\partial/\partial\phi_2$ -axis. Thus, we see that the rod structure enters the reduced field equations through the boundary conditions imposed upon the matrix field G_{ij} . The horizon topology is also determined by

⁷In [14], neither the condition that v^1, v^2 be integers, nor the determinant conditions for adjacent rod vectors and their relation to the horizon topology were obtained. Furthermore, his rod vectors have 3 components, rather than 2.

the rod structure by Proposition 2, see also Remark 2 following that proposition. This is how the different topology and global nature of the solutions in 5 dimensions are encoded in the reduced Einstein equations on the upper half plane \hat{M} .

Clearly, since we have argued that the rod structure is a diffeomorphism invariant datum constructed from the given solution, two given stationary black hole solutions with 2 axial Killing fields cannot be isomorphic unless the rod structures and the masses and angular momenta coincide. The main purpose of this paper is to point out the following converse to this statement:

Theorem: Consider two stationary, asymptotically flat, vacuum black hole spacetimes of dimension 5, having two commuting axial Killing fields that commute also with the time-translation Killing field. Assume that both solutions have the same rod structure, and the same values of the mass m and angular momenta J_1, J_2 . Then they are isometric.

Proof: As in 4 spacetime dimensions, the key step in the argument is to put the reduced Einstein equations in a suitable form. Following [26] (see also [25]), this is done as follows in 5 dimensions. On M , we first define the two twist 1-forms

$$\omega_{1a} = \varepsilon_{abcde} \psi_1^b \psi_2^c \nabla^d \psi_1^e \quad (42)$$

$$\omega_{2a} = \varepsilon_{abcde} \psi_1^b \psi_2^c \nabla^d \psi_2^e. \quad (43)$$

Using the vacuum field equations and standard identities for Killing fields [33], one shows that these 1-forms are closed. Since the Killing fields commute, the twist forms are invariant under \mathcal{G} , and so we may define corresponding 1-forms $\hat{\omega}_{1a}$ and $\hat{\omega}_{2a}$ on the interior of the factor space $\hat{M} = \{\zeta \in \mathbb{C}; \text{Im} \zeta > 0\}$. These 1-forms are again closed. Thus, the “twist potentials”

$$\chi_i = \int_0^\zeta \hat{\omega}_{i\zeta} d\zeta + \hat{\omega}_{i\bar{\zeta}} d\bar{\zeta} \quad (44)$$

are globally defined on \hat{M} and independent of the path connecting 0 and ζ . We introduce the 3×3 matrix field Φ by

$$\Phi = \begin{pmatrix} (\det f)^{-1} & -(\det f)^{-1} \chi_1 & -(\det f)^{-1} \chi_2 \\ -(\det f)^{-1} \chi_1 & f_{11} + (\det f)^{-1} \chi_1 \chi_1 & f_{12} + (\det f)^{-1} \chi_1 \chi_2 \\ -(\det f)^{-1} \chi_2 & f_{21} + (\det f)^{-1} \chi_2 \chi_1 & f_{22} + (\det f)^{-1} \chi_2 \chi_2 \end{pmatrix}. \quad (45)$$

Here f_{ij} is the Gram matrix of the axial Killing fields,

$$f = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}. \quad (46)$$

The matrix Φ satisfies $\Phi^T = \Phi$, $\det \Phi = 1$, and is positive semi-definite, meaning that it may be written in the form $\Phi = S^T S$ for some matrix S of determinant 1. As a consequence of the reduced Einstein equations (4) and (5), it also satisfies the divergence identity

$$\hat{D}^a [r \Phi^{-1} \hat{D}_a \Phi] = 0 \quad (47)$$

on \hat{M} .

Consider now the exterior of the two black hole solutions as in the statement of the theorem, denoted (M, g_{ab}) and $(\tilde{M}, \tilde{g}_{ab})$. We denote the corresponding matrices defined as above by Φ and $\tilde{\Phi}$, and we use the same “tilde” notation to distinguish any other quantities associated with the two solutions. Since the orbit spaces of the respective spacetimes can both be identified with the upper half-plane as analytic manifolds, we can identify the respective orbit spaces. Furthermore, one can show by reversing the constructions of the local analytic coordinate systems in the proof of Proposition 1 that the \mathcal{G} -manifold M can be uniquely reconstructed from the rod structure, i.e., M as a manifold with a \mathcal{G} -action is uniquely determined by the rod structure modulo diffeomorphisms preserving the action of \mathcal{G} . Therefore, since the rod structures $\{\tilde{l}_j, \tilde{\mathbf{v}}_j\}$ and $\{l_j, \mathbf{v}_j\}$ are the same, M and \tilde{M} are isomorphic as manifolds with a \mathcal{G} action, and we may hence assume that $M = \tilde{M}$, and that $\tilde{t}^a = t^a$, $\tilde{\Psi}_i^a = \Psi_i^a$ for $i = 1, 2$. It follows in particular that \tilde{g}_{ab} and g_{ab} may be viewed as being defined on the same analytic manifold, M , and we may also assume that $\tilde{r} = r$ and $\tilde{z} = z$. Consequently, it is possible to combine the divergence identities (47) for the two solutions into a single identity on the upper complex half plane. This key identity [27, 25] is called the “Mazur identity” and is given by:

$$\hat{D}_a(r\hat{D}^a\text{Tr}\Psi) = r\hat{g}^{ab}\text{Tr}[\Delta g_a^T\tilde{\Phi}\Delta g_b\Phi^{-1}] \quad (48)$$

where

$$\Psi = \tilde{\Phi}\Phi^{-1} - 1, \quad \Delta g_a = \tilde{\Phi}^{-1}\hat{D}_a\tilde{\Phi} - \Phi^{-1}\hat{D}_a\Phi. \quad (49)$$

Using now the identities $\Phi = S^T S$ and $\tilde{\Phi} = \tilde{S}^T \tilde{S}$, the Mazur identity can be presented in the form

$$\hat{D}_a(r\hat{D}^a\text{Tr}\Psi) = r\hat{g}^{ab}\text{Tr}[N_a^T N_b] \quad (50)$$

where $N_a = S^{-1}\Delta g_a\tilde{S}$.

The key point about the Mazur identity (50) is that on the left side we have a total divergence, while the term on the right hand side is non-negative. This structure is now exploited by integrating the Mazur identity over \hat{M} . Using Gauss’ theorem, one finds

$$\int_{\partial\hat{M}\cup\infty} r\hat{D}_a\text{Tr}\Psi d\hat{S}^a = \int_{\hat{M}} r\hat{g}^{ab}\text{Tr}[N_a^T N_b] d\hat{V}, \quad (51)$$

where the integral over the boundary includes an integration over the “boundary at infinity”. If one can show that the boundary integral on the left side is zero, then it follows that N^a vanishes on \hat{M} , and hence that $\tilde{\Phi}^{-1}\hat{D}_a\tilde{\Phi} = \Phi^{-1}\hat{D}_a\Phi$. Since this implies that $\Phi^{-1}\tilde{\Phi}$ is a constant matrix on \hat{M} , one concludes that $\Phi = \tilde{\Phi}$ if this holds true at one point of \hat{M} . Using that the Gram matrices \tilde{f}_{ij} and f_{ij} become equal near infinity, and using that $\tilde{\chi}_i$ is equal to χ_i on the axis (see below), one concludes that Φ is equal to $\tilde{\Phi}$ on an axis near infinity, and hence equal everywhere in \hat{M} .

This can now be used as follows to show that the spacetimes are isometric. First, it immediately follows from $\tilde{\Phi} = \Phi$ that $\tilde{\chi}_i = \chi_i$ and that the Gram matrices of the axial Killing fields are identical for the two solutions, $\tilde{f}_{ij} = f_{ij}$. To see that the other scalar products between the Killing fields coincide for the two solutions, let $\alpha_i = g_{ab}t^a\Psi_i^b$, $\beta = g_{ab}t^a t^b$, and define similarly the scalar products $\tilde{\alpha}_i, \tilde{\beta}$ for the other spacetime. One derives the equation

$$\hat{D}_a[(f^{-1})^{ij}\alpha_j] = r(\det f)^{-1}\hat{\epsilon}_a{}^b(f^{-1})^{ij}\hat{D}_b\chi_j. \quad (52)$$

The right side does not depend upon the conformal factor v , so since $\tilde{\chi}_i = \chi_i$ and $\tilde{f}_{ij} = f_{ij}$, it also follows that $\tilde{\alpha}_i = \alpha_i$ up to a constant. That constant has to vanish, since it vanishes at infinity. Furthermore, from

$$\beta = (f^{-1})^{ij}\alpha_i\alpha_j - (\det f)^{-1}r^2 \quad (53)$$

we have $\tilde{\beta} = \beta$. Thus, all scalar products of the Killing fields are equal for the two solutions, $\tilde{G}_{IJ} = G_{IJ}$ on the entire upper half plane. Viewing now the reduced Einstein equation (5) as an equation for v respectively \tilde{v} , one concludes from this that $\tilde{v} = v$. Thus, summarizing, we have shown that if the boundary integral in the integrated Mazur identity eq. (51) could be shown to vanish, then $\tilde{G}_{IJ} = G_{IJ}$, $\tilde{r} = r$, $\tilde{z} = z$ and $\tilde{v} = v$. Since $\tilde{t}^a = t^a$, $\tilde{\Psi}_i^a = \Psi_i^a$ it follows from eqs. (36) and (37) that $\tilde{g}_{ab} = g_{ab}$. This proves that the two spacetimes are isometric, proving the theorem.

Thus, to establish the statement of the theorem, one needs to prove that the boundary integral in (51) vanishes. For this, one has to analyze the behavior of the integrand $r\hat{D}_a\text{Tr}\Psi$ near the boundary $\text{Im}\zeta = 0$ (i.e., the horizon and the axis) and near the boundary at infinity, $\text{Im}\zeta \rightarrow \infty$ as $\text{Re}\zeta$ is kept fixed (i.e., spatial infinity). At this stage one has to use again that the asymptotic charges and the rod structure of the solutions are assumed to be identical. We divide the boundary region into three parts: (1) The axis, (2) the horizon, and (3) infinity.

- (1) On each segment (z_j, z_{j+1}) of the real line $\text{Im}\zeta = r = 0$ representing an axis, we know that the null spaces of the Gram matrices f_{ij} and \tilde{f}_{ij} coincide, because we are assuming that the rod structures of both solutions are identical. Furthermore, from eq. (44), and from the fact that $\hat{\omega}_a^i$ vanishes on any axis by definition, the twist potentials χ_i are constant on the real line outside of the segment (z_h, z_{h+1}) representing the horizon. The difference between the constant value of χ_i on the real line left and right to the horizon segment can be calculated as follows:

$$\begin{aligned} \chi_i(r=0, z_h) - \chi_i(r=0, z_{h+1}) &= \int_{z_h}^{z_{h+1}} \hat{\omega}_i^\zeta d\zeta + \hat{\omega}_i^{\bar{\zeta}} d\bar{\zeta} \\ &= \frac{1}{(2\pi)^2} \int_{\mathcal{H}} \nabla_{[a}\Psi_{b]i} dS^{ab} \\ &= \frac{1}{(2\pi)^2} \int_{S^3_\infty} \nabla_{[a}\Psi_{b]i} dS^{ab} = \text{const. } J_i. \end{aligned}$$

The first equality follows from the definition of the twist potentials, the second from the defining formula for the twist potentials and the fact that the twist potentials are invariant under the action of the 2-independent rotation isometries each with period 2π (with \mathcal{H} a horizon cross section), the third equation follows from Gauss' theorem and the fact that $\nabla^a(\nabla_{[a}\Psi_{b]i}) = 0$ when $R_{ab} = 0$, and the last equality follows from the Komar expression for the angular momentum in 5 dimensions. The analogous expressions hold in the spacetime $(\tilde{M}, \tilde{g}_{ab})$. Because we assume that $J_i = \tilde{J}_i$, we can add constants to χ_i , if necessary, so that $\chi_i = \tilde{\chi}_i$ on the axis, and in fact that $\Delta\chi_i = \chi_i - \tilde{\chi}_i = O(r^2)$ near any axis. One may now analyze the contributions to the boundary integral coming from the axis using the expression

$$\hat{D}_a\text{Tr}\Psi = \hat{D}_a \left[(\det \tilde{f})^{-1} \{-\Delta(\det f) + (f^{-1})^{ij}\Delta\chi_i\Delta\chi_j\} + (f^{-1})^{ij}\Delta f_{ij} \right]. \quad (54)$$

We consider a particular axis rod with rod vector $\mathbf{v} = (v^1, v^2)$, which by assumption is identical for the two solutions. We pick a second basis vector $\mathbf{w} = (w^1, w^2)$, and we denote by $\mathbf{v}^*, \mathbf{w}^*$ the dual basis. We conclude that, on the given rod

$$f_{ij} = av_i^* v_j^* + bw_i^* w_j^* + 2cv_{(i}^* w_{j)}^*, \quad \tilde{f}_{ij} = \tilde{a}v_i^* v_j^* + \tilde{b}w_i^* w_j^* + 2\tilde{c}v_{(i}^* w_{j)}^*, \quad (55)$$

with $\tilde{a} = O(r^2) = a, \tilde{b} = O(1) = b$ and $\tilde{c} = O(r) = c$. We insert this into eq. (54), we use that $\Delta\chi_i = O(r^2)$ on the axis, and we use the detailed fall-off properties of the metric as well as the quantities $\chi_i, f_{ij}, \tilde{\chi}_i, \tilde{f}_{ij}$ for large z , which are the same for any asymptotically flat solution to the relevant order. One finds that $\hat{D}_a \text{Tr} \Psi$ is finite on the axis, so that the corresponding contribution to the line integral vanishes. The details of this analysis are in close parallel to the corresponding analysis of Ida et al. [26], who analyzed the situation for a special horizon topology and rod structure.

- (2) On the horizon segment, the matrices f_{ij}, \tilde{f}_{ij} are invertible, so $\hat{D}_a \text{Tr} \Psi$ is regular, and the boundary integral over this segment vanishes.
- (3) Near infinity, one has to use the asymptotic form of the metric for an asymptotically flat spacetime in 5 dimensions. In an appropriate asymptotically Minkowskian coordinate system such that asymptotically $\psi_1^a = (\partial/\partial\phi_1)^a$ and $\psi_2^a = (\partial/\partial\phi_2)^a$, it takes the form

$$\begin{aligned} g = & - \left(1 - \frac{\mu}{R^2} + O(R^{-2}) \right) dt^2 + \left(\frac{2\mu a_1 (R^2 + a_1^2)}{R^4} \sin^2 \theta + O(R^{-3}) \right) dt d\phi_1 \\ & + \left(\frac{2\mu a_2 (R^2 + a_2^2)}{R^4} \cos^2 \theta + O(R^{-3}) \right) dt d\phi_2 + \left(1 + \frac{\mu}{2R^2} + O(R^{-3}) \right) \times \\ & \times \left(\frac{R^2 + a_1^2 \cos^2 \theta + a_2^2 \sin^2 \theta}{(R^2 + a_1^2)(R^2 + a_2^2)} R^2 dR^2 + (R^2 + a_1^2 \cos^2 \theta + a_2^2 \sin^2 \theta) d\theta^2 \right. \\ & \left. + (R^2 + a_1^2) \sin^2 \theta d\phi_1^2 + (R^2 + a_2^2) \cos^2 \theta d\phi_2^2 \right) \end{aligned}$$

where μ, a_1, a_2 are parameters proportional to the mass, and the angular momenta J_1, J_2 of the solution. One must then determine the functions z, r as functions of R, θ near infinity using the reduced Einstein equations, subject to the boundary condition (39) near infinity. This then enables one to find asymptotic expansions for $f_{ij}, \tilde{f}_{ij}, \chi_i, \tilde{\chi}_i$ in terms of the parameters $J_1, J_2, m, \tilde{J}_1, \tilde{J}_2, \tilde{m}$. Using that these parameters coincide for both solutions, one shows that the contribution to the boundary integral (50) vanishes. Again, the details of this argument only depend upon the asymptotics of the solution, but not the horizon topology, or rod structure. They are therefore identical to the arguments given in [26] for spherical black holes, see also [14, Sec. 4.3].

This completes the proof. \square

5 Conclusion

In this paper we have considered 5-dimensional stationary, asymptotically flat, vacuum black hole spacetimes with 2 commuting axial Killing fields generating an action of $U(1) \times U(1)$.

Under the additional hypothesis that there are no points with a discrete isotropy subgroup, we have shown that the black hole must have horizon topology $S^3, S^2 \times S^1$, or $L(p, q)$, and that each solution is uniquely specified by the asymptotic charges (mass and the two angular momenta), together with certain data describing the relative position and distance of the horizon and axis of rotation—the “rod-structure,” defined in a somewhat different form first by [14]. Our proof mostly relied on the known σ -model formulation of the reduced Einstein equation [25, 26], combined with basic arguments clarifying the global structure of the factor manifold of symmetry orbits.

As we have already pointed out in the introduction, the case considered in this paper presumably does not represent the most general stationary, asymptotically flat black hole solution in 5 dimensions. It appears highly unlikely that our method of proof could be generalized to solutions with only one axial Killing field, if such solutions were to exist. On the other hand, we believe that our assumption that there are no points with a discrete isotropy group is only of a technical nature. Without this assumption, the orbit space will have singular points (“orbifold points”), rather than being an analytic manifold with boundary. Our analysis of the integrated divergence identity (51) then would also have to include the boundaries resulting from the blow ups of the orbifold points. It seems not unlikely that the proof could still go through if the nature of the discrete subgroups is identical for the two solutions. Thus, it appears that we need to specify in general at least (a) the mass and angular momenta (b) rod structure, and (c) a datum describing the position of the points with discrete isotropy subgroups in the upper half plane, together with the specification of the subgroups themselves.

It is also interesting to ask how the parameters in the rod structure are related to other properties of the solution, such as the invariant charges, horizon area, or surface gravity. For example, by evaluating the horizon area for the metric (36), one finds that the rod parameter l_h associated with the horizon is given by $l_h = \kappa A / 4\pi^2$, but we do not know whether similar relations exist for the other rod parameters. It is also not clear whether all rod structures can actually be realized in solutions to the vacuum equations, nor whether the horizon topologies $L(p, q)$ can be realized⁸. Finally it would be interesting to see if the constructions of this paper can be generalized to include matter fields [20].

Acknowledgements: We would like to thank Troels Harmark for comments and Thomas Schick for discussions. S. Y. would like to thank the Alexander von Humboldt Foundation for a stipend, and the Institut für Theoretische Physik Göttingen for its kind hospitality. He also acknowledges financial support from the Bulgarian National Science Fund under Grant MUF04/05 (MU 408).

Note added in proof: After this manuscript was posted, it was noted by P. Chrusciel that our analysis did not properly take into account points with discrete isotropy group. We have added a corresponding assumption to the hypothesis. We are grateful to him for sharing his insight with us.

⁸Solutions with horizon topology $L(n, 1)$ have however been found in Einstein-Maxwell theory, see [21].

References

- [1] Adams, C.C.: *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots* New York, W. H. Freeman (1994)
- [2] Bunting, G. L.: Proof of the uniqueness conjecture for black holes, (PhD Thesis, Univ. of New England, Armidale, N.S.W., 1983)
- [3] Carter, B.: Axisymmetric black hole has only two degrees of freedom, *Phys. Rev. Lett.* **26**, 331-333 (1971)
- [4] Y. M. Cho and P. G. O. Freund: Non-Abelian gauge fields as Nambu-Goldstone fields, *Phys. Rev. D* **12**, 1711 (1975)
- [5] Chruściel, P.T.: On rigidity of analytic black holes, *Commun. Math. Phys.* **189**, 1-7 (1997)
- [6] Emparan, R. and Reall, H. S.: A rotating black ring in five dimensions. *Phys. Rev. Lett.* **88**, 101101 (2002)
- [7] Emparan, R. and Reall, H. S.: Generalized Weyl solutions, *Phys. Rev. D* **65**, 084025 (2002)
- [8] Friedrich, H., Racz, I. and Wald, R.M.: On the rigidity theorem for spacetimes with a stationary event horizon or a compact Cauchy horizon, *Commun. Math. Phys.* **204**, 691-707 (1999)
- [9] Galloway, G. J., Schleich, K., Witt, D. M., and Woolgar, E.: Topological censorship and higher genus black holes, *Phys. Rev. D* **60**, 104039 (1999)
- [10] Galloway, G. J., Schleich, K., Witt, D., and Woolgar, E.: The AdS/CFT correspondence conjecture and topological censorship, *Phys. Lett. B* **505**, 255 (2001)
- [11] Galloway, G.J., Schoen, R.: A Generalization of Hawking's black hole topology theorem to higher dimensions, *Commun. Math. Phys.* **266**, 571 (2006)
- [12] Gibbons G. W., Ida, D., and Shiromizu, T.: Uniqueness and non-uniqueness of static black holes in higher dimensions, *Phys. Rev. Lett.* **89**, 041101 (2002)
- [13] Harmark T. and Olesen, P: On the structure of stationary and axisymmetric metrics, *Phys. Rev. D* **72**, 124017 (2005) [arXiv:hep-th/0508208].
- [14] Harmark, T.: Stationary and axisymmetric solutions of higher-dimensional general relativity, *Phys. Rev. D* **70**, 124002 (2004) [arXiv:hep-th/0408141].
- [15] Hawking, S.W.: Black holes in general relativity. *Commun. Math. Phys.* **25**, 152-166 (1972)
- [16] Hawking, S.W. and Ellis, G.F.R.: *The large scale structure of space-time* Cambridge: Cambridge University Press, 1973

- [17] Heusler, M.: *Black hole uniqueness theorems*, Cambridge University Press (1996)
- [18] Hollands, S., Ishibashi, A. and Wald, R. M.: A higher dimensional stationary rotating black hole must be axisymmetric, *Commun. Math. Phys.* **271**, 699 (2007) [arXiv:gr-qc/0605106].
- [19] Hollands, S. and Ishibashi, A.: Asymptotic flatness and Bondi energy in higher dimensional gravity, *J. Math. Phys.* **46**, 022503 (2005) [arXiv:gr-qc/0304054].
- [20] Hollands, S. and Yazadjiev, S., Work in progress
- [21] Ishihara, H., Kimura, M., Masuno, K., Tomizawa, S.: Black holes on Euguchi-Hanson space in five-dimensional Einstein Maxwell theory, *Phys. Rev. D* **74**, 047501 (2006)
- [22] Israel, W.: Event horizons in static vacuum space-times, *Phys. Rev.*, **164**, 1776-1779 (1967)
- [23] R. Kerner: Generalization of Kaluza-Klein theory for an arbitrary non-abelian gauge group, *Ann. Inst. H. Poincare*, **9**, 143 (1968)
- [24] Kobayashi, S., Nomizu, K.: *Foundations of Differential Geometry I*, Wiley 1969
- [25] Maison, D.: Ehlers-Harrison-type Transformations for Jordan's extended theory of gravitation, *Gen. Rel. Grav.* **10**, 717 (1979)
- [26] Morisawa, Y., Ida, D.: A boundary value problem for five-dimensional stationary black holes, *Phys. Rev. D* **69**, 124005 (2004)
- [27] Mazur, P. O.: Proof of uniqueness of the Kerr-Newman black hole solution, *J. Phys. A*, **15**, 3173-3180 (1982)
- [28] Moncrief, V. and Isenberg, J.: Symmetries of cosmological Cauchy horizons. *Commun. Math. Phys.* **89**, 387-413 (1983)
- [29] Myers, R.C. and Perry, M.J.: Black holes in higher dimensional space-times. *Annals Phys.* **172** 304 (1986)
- [30] Racz, I.: On further generalization of the rigidity theorem for spacetimes with a stationary event horizon or a compact Cauchy horizon. *Class. Quant. Grav.* **17**, 153 (2000)
- [31] Robinson, D. C.: Uniqueness of the Kerr black hole, *Phys. Rev. Lett.* **34**, 905-906 (1975)
- [32] Sudarsky, D. and Wald, R.M.: Extrema of mass, stationarity, and staticity, and solutions to the Einstein Yang-Mills equations. *Phys. Rev. D* **46** 1453-1474 (1992)
- [33] Wald, R.M.: *General Relativity*. Chicago: University of Chicago Press, 1984
- [34] Weinstein, G.: On rotating black holes in equilibrium in general relativity, *Commun. Pure Appl. Math.* **43**, 903 (1990)