Integer Polynomial Optimization

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This talk deals with the problem of optimizing polynomial functions over the lattice points in a polyhedron when the number of variables is a constant.

We explain why the problem is already hard in dimension two for polynomial functions of degree four. Then we will discuss how to solve the problem in polynomial time when the function is a quadratic polynomial in two variables.

Further complexity results about optimizing homogeneous polynomials and cubic polynomials over the integer points in polyhedra in dimension two will be presented too.

In arbitrary but fixed dimension the optimization of a polynomial over the lattice points in a polyhedron remains computationally challenging.

When the polynomial is positive over a polytope, we show that there exists an FPTAS for approximating its maximal value that is based on Barvinok's algorithm for counting the number of lattice points in polyhedra. In order to develop an FPTAS for further classes of nonlinear functions to be minimized over integer points in polyhedra, we propose a framework that combines the techniques of Papadimitriou and Yannakakis with ideas similar to those commonly used to derive certificates of positivity for polynomials over semialgebraic sets. Generally speaking, we work with classes of "basic functions". Then, for a given f, we try to detect a decomposition of f as a finite sum of products of a so-called "sliceable function" and a basic function f_i. Roughly speaking, sliceable functions can be approximated by subdividing the given polyhedron. For instance, the set of all convex functions presented by a first order oracle that are nonnegative over a given polytope could serve as a class of basic functions, because we can solve the problem for any member in the class in polynomial time when n is fixed. Our technique allows us to prove the following result.

Let Q be a symmetric matrix with integer coefficients and let n be fixed. Then there is an FPTAS for minimizing $f(x) = x^T Q x$ over the integer vectors x in a polytope in the following cases:

1. Q has at most one negative eigenvalue;

2. Q has at most one positive eigenvalue.

The talk is based on four papers that emerged from joint work with the following people: Robert Hildebrand, Raymond Hemmecke, Matthias Köppe, Alberto del Pia, Jesus de Loera and Kevin Zemmer.